On Ultra-Short Block-Codes on the Binary Asymmetric Channel

Po-Ning Chen, Hsuan-Yin Lin, Stefan M. Moser
Department of Electrical Engineering
National Chiao Tung University (NCTU)
Hsinchu 30010, Taiwan
qponing@mail.nctu.edu.tw, {lin.hsuanin,stefan.moser}@ieee.org

1 Introduction

Shannon proved in his ground-breaking work that it is possible to find an information transmission scheme that can transmit messages at arbitrarily small error probability as long as the transmission rate in bits per channel use is below the so-called capacity of the channel and the blocklength is very large. However, a large blocklength might not be an option and might have to be restricted to some reasonable size. The question now arises what we can theoretically say about the performance of communication systems with such restricted blocksize and about good design for such codes.

To simplify the problem, we currently focus on binary channels and start with the simplest type of communication: a code with only two possible, equally likely messages.

2 Channel Model

We consider a general binary channel binary asymmetric channel (BAC) with crossover probabilities that are not identical:

\[
\begin{array}{ccc}
0 & 1 - \epsilon_0 & 0 \\
\epsilon_1 & X & Y \\
1 & 1 - \epsilon_1 & 1
\end{array}
\]

Without loss of generality we can restrict the values of these parameters as follows:

\[0 \leq \epsilon_0 \leq \epsilon_1 \leq 1, \quad \epsilon_0 \leq 1 - \epsilon_0, \quad \epsilon_0 \leq 1 - \epsilon_1.\]

3 Main Results

Definition 1. We define the flip-flop code of type \( t \) as follows: for every \( t \in \{0, 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \} \), we have

\[C_t = \left( \frac{x_1}{x_2} \right) \triangleq \left( \frac{x}{\bar{x}} \right)\]

where

\[x_1 = x \triangleq 00 \cdots 0 11 \cdots 1, \quad \text{w}_H(x) = t\]

\[x_2 = \bar{x} \triangleq 11 \cdots 1 00 \cdots 0\]

Proposition 2. Fix the blocklength \( n \). Then, irrespective of the BAC channel parameters \( \epsilon_0 \) and \( \epsilon_1 \), there always exists a flip-flop code of type \( t \), \( C_t \), for some choice of \( 0 \leq t \leq \left\lfloor \frac{n}{2} \right\rfloor \) that is optimal in the sense that it minimizes the error probability.

In the situation of a flip-flop code of type \( t \), \( C_t \), the log likelihood ratio \( \text{LLR}_t^{(n)}(\epsilon_0, \epsilon_1, d) \) is a function of the BAC channel parameters and of \( d \), the Hamming distance between the received vector and the first codeword.

Proposition 3. For a fixed flip-flop code \( C_t^{(n)} \) and a fixed BAC \((\epsilon_0, \epsilon_1)\), there exists a threshold \( \ell \), \( t \leq \ell \leq \left\lfloor \frac{n-1}{2} \right\rfloor \), such that the optimal ML decision rule can be stated as

\[g(y) = \begin{cases} x_1 & \text{if } 0 \leq d \leq \ell, \\ x_2 & \text{if } \ell + 1 \leq d \leq n. \end{cases}\]

Theorem 4. Fix blocklength \( n \). Under a particular fixed decision rule \( \ell \), the flip-flop codebook of type \( t \) is optimal if \( (\epsilon_0, \epsilon_1) \) belongs to

\[\left\{ (\epsilon_0, \epsilon_1) \left| \text{LLR}_t^{(n-1)}(\epsilon_0, \epsilon_1, \ell) > 0 \quad \land \quad \text{LLR}_{t+1}^{(n-1)}(\epsilon_0, \epsilon_1, \ell) < 0 \right. \right\}.

If the region is empty, then \( t \) is not optimal for any BAC.

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