On Ultra-Short Block-Codes on Two Special Binary Channels

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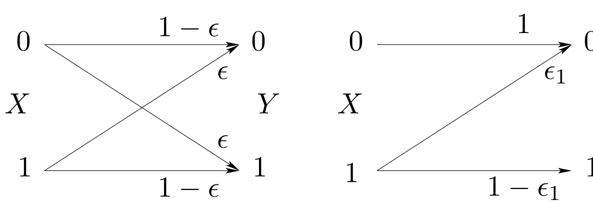
1 Introduction

Shannon proved in his ground-breaking work that it is possible to find an information transmission scheme that can transmit messages at arbitrarily small error probability as long as the transmission rate in **bits per channel use** is below the so-called **capacity** of the channel and the blocklength is very large. However, a large blocklength might not be practical. **The question now arises what we can theoretically say about the performance of communication systems with strongly restricted blocklength and their optimal design.**

Here we focus on the number of messages up to at most four. Our goal is to try to find the optimal code structure with respect to the **average error probability for any fixed blocklength n** on the binary symmetric channel (BSC) and the Z-channel. I.e., for a fixed n we try to design a code that minimizes the error probability among all possible codes of length n.

2 Channel Models

We consider two well-known binary channels: the **binary symmetric channel (BSC)** with crossover probability ϵ and the **Z-channel** with 1–0-crossover probability ϵ_1:



BSC Z-channel

3 Optimal Code Design

Definition 1. *The **flip code of type** t for $t \in \left\{0, 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\}$ is defined by the following codebook matrix $\mathcal{C}$ with $\mathcal{M} = 2$ codewords:*

$$\mathcal{C}^{(2)} = \begin{pmatrix} 0 & \cdots & 0 & \overbrace{1 & \cdots & 1}^{t} \\ 1 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix}$$

*A **weakly flip code of length** n for $\mathcal{M} = 3$ or $\mathcal{M} = 4$ codewords is defined by a codebook matrix that consists of n columns taken from the following candidate sets:*

$$\left\{ \mathbf{c}_1^{(3)} \triangleq \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \mathbf{c}_2^{(3)} \triangleq \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{c}_3^{(3)} \triangleq \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

or

$$\left\{ \mathbf{c}_1^{(4)} \triangleq \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \mathbf{c}_2^{(4)} \triangleq \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{c}_3^{(4)} \triangleq \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\},$$

respectively.

Proposition 2. *For $\mathcal{M} = 2$ codewords and any blocklength n, on a BSC, the **flip codes of type** t for any $t \in \left\{0, 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\}$ are all optimal, and on a Z-channel, the **flip code of type** 0 is optimal.*

Theorem 3. *For $\mathcal{M} = 3$ or $\mathcal{M} = 4$ codewords and any blocklength n, on a BSC, an optimal code is the **weakly flip code** constructed recursively by the following choice of candidate columns:*

$$\mathcal{C}^{(\mathcal{M})} = \left(\mathbf{c}_1^{(\mathcal{M})}\, \mathbf{c}_3^{(\mathcal{M})}\, \mathbf{c}_1^{(\mathcal{M})}\, \mathbf{c}_2^{(\mathcal{M})}\, \mathbf{c}_3^{(\mathcal{M})}\, \mathbf{c}_1^{(\mathcal{M})}\, \mathbf{c}_2^{(\mathcal{M})}\, \cdots \right).$$

On a Z-channel, an optimal choice for the codebook matrix is

$$\mathcal{C}^{(\mathcal{M})} = \left(\mathbf{c}_1^{(\mathcal{M})}\, \mathbf{c}_2^{(\mathcal{M})}\, \mathbf{c}_1^{(\mathcal{M})}\, \mathbf{c}_2^{(\mathcal{M})}\, \mathbf{c}_1^{(\mathcal{M})}\, \mathbf{c}_2^{(\mathcal{M})}\, \cdots \right).$$

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