On the Suboptimality of Equidistant Codes Meeting the Plotkin Bound

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1 Fair Weak Flip Codes

In this poster, we re-introduce from our previous work a new family of **nonlinear** codes: **fair weak flip codes**. They belong to the class of **equidistant codes**, i.e., they satisfy that any two distinct codewords have identical Hamming distance.

In the case of $\mathrm{M} = 5, 6$, we define the following fair weak flip codes:

$$\mathscr{C}_{\text{fair}}^{(5,10)} \triangleq \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&1&1&1&1&1\\ 0&1&1&1&0&0&0&1&1&1\\ 1&0&1&1&0&1&1&0&0&1\\ 1&1&0&1&1&0&1&0&1&0 \end{pmatrix}$$

$$\mathscr{C}_{\text{fair}}^{(6,10)} \triangleq \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&1&1&1&1&1\\ 0&1&1&1&0&0&0&1&1&1\\ 1&0&1&1&0&1&1&0&0&1\\ 1&1&0&1&1&0&1&0&1&0\\ 1&1&1&0&1&1&0&1&0&0 \end{pmatrix}$$

Both of them have the following properties:

- Each column's first component is 0 and its Hamming weight equals to $\lfloor \frac{\mathrm{M}}{2} \rfloor$ or $\lceil \frac{\mathrm{M}}{2} \rceil$.
- They are called *fair* since it is constructed by an **equal number of all possible** such columns (the number is called L).
- Each fair code can be constructed by duplicating $\mathscr{C}_{\text{fair}}^{(\mathrm{M},\mathrm{L})}$ many times.
- The fair weak flip codes have a maximum minimum Hamming distance and achieve the Plotkin bound.
- These codes are **not optimal in the sense of average error probability over the binary symmetric channel (BSC)**.

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2 Main Results

Proposition: Consider a BSC of conditional channel probability

$$P_{Y|X}(y|x) = \begin{cases} 1-\epsilon & \text{if } y = x, \\ \epsilon & \text{if } y \neq x, \end{cases} \qquad x, y \in \{0,1\}$$

with crossover probability $0 < \epsilon < \frac{1}{2}$. For a fair weak flip code $\mathscr{C}_{\text{fair}}^{(\mathrm{M},n)}$ with a corresponding blocklength, let $\mathscr{C}_{\text{reduced}}^{(\mathrm{M},n-1)}$ be a code that is created from $\mathscr{C}_{\text{fair}}^{(\mathrm{M},n)}$ by deleting an arbitrary column in the codebook matrix. Then

$$P_{\text{c}}\Big(\mathscr{C}_{\text{fair}}^{(\mathrm{M},n)}\Big) = P_{\text{c}}\Big(\mathscr{C}_{\text{reduced}}^{(\mathrm{M},n-1)}\Big)$$

Moreover, let $\mathscr{C}_{\text{unfair}}^{(\mathrm{M},n)}$ be a code that is created by appending a weak flip column to $\mathscr{C}_{\text{reduced}}^{(\mathrm{M},n-1)}$ such that it is not a fair weak flip code. Then

$$P_{\text{c}}\Big(\mathscr{C}_{\text{unfair}}^{(\mathrm{M},n)}\Big) > P_{\text{c}}\Big(\mathscr{C}_{\text{reduced}}^{(\mathrm{M},n-1)}\Big)$$

Theorem: **Fair weak flip codes** with an arbitrary number of codewords M and with a blocklength n such that $n \bmod \mathrm{L} = 0$ are **strictly suboptimal on a BSC**.

References

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[2] P.-N. Chen, H.-Y. Lin, and S. M. Moser, "Optimal ultrasmall block-codes for binary discrete memoryless channels," IEEE Trans. Inf. Theory, vol. 59, no. 11, pp. 73467378, Nov. 2013.