Optimal Ultra-Small Block-Codes on Two Special Binary Channels

Po-Ning Chen, Hsuan-Yin Lin, Stefan M. Moser
Department of Electrical Engineering
National Chiao Tung University (NCTU)
Hsinchu 30010, Taiwan
qponing@mail.nctu.edu.tw, {lin.hsuanyin, stefan.moser}@ieee.org

1 Introduction

Shannon proved in his ground-breaking work that it is possible to find an information transmission scheme that can transmit messages at arbitrarily small error probability as long as the transmission rate in bits per channel use is below the so-called capacity of the channel and the blocklength is very large. However, a large blocklength might not be practical. The question now arises what we can theoretically say about the performance of communication systems with strongly restricted blocklength and their global optimal design.

Here we focus on the number of messages up to at most four. Our goal is to try to find the optimal code structure (either linear or nonlinear) with respect to the average error probability for any fixed blocklength \( n \) on the binary symmetric channel (BSC) and the Z-channel. I.e., for a fixed \( n \) we try to design a code that minimizes the error probability among all possible codes of length \( n \).

2 Channel Models

We consider two well-known binary channels: the binary symmetric channel (BSC) with crossover probability \( \epsilon \) and the Z-channel with 1–0-crossover probability \( \epsilon_1 \):

\[
\begin{array}{c|c|c}
X & 0 & 1 \\
\hline
\epsilon & 1 & \epsilon_1 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
1 & 0 \\
\hline
1 - \epsilon & 1 & 1 - \epsilon_1 \\
\end{array}
\]

Z-channel

\[
\begin{array}{c|c|c}
0 & 1 - \epsilon & 0 \\
\hline
\epsilon & 1 & \epsilon_1 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
X & 0 & 1 \\
\hline
\epsilon & 1 & \epsilon_1 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
1 & 0 \\
\hline
1 - \epsilon & 1 & 1 - \epsilon_1 \\
\end{array}
\]

BSC

3 Optimal Code Design

Definition 1. The flip code of type \( t \) for \( t \in \{0, 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \} \) is defined by the following codebook matrix \( \mathcal{C} \) with \( M = 2 \) codewords:

\[
\mathcal{C}^{(2)} = \begin{pmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \end{pmatrix}
\]

A weakly flip code of length \( n \) for \( M = 3 \) or \( M = 4 \) codewords is defined by a codebook matrix that consists of \( n \) columns taken from the following candidate sets:

\[
\begin{cases}
\{c_1^{(3)} \equiv 0, c_2^{(3)} \equiv 0, c_3^{(3)} \equiv 1, c_4^{(3)} \equiv 1\} \\
\{c_1^{(4)} \equiv 0, c_2^{(4)} \equiv 0, c_3^{(4)} \equiv 1, c_4^{(4)} \equiv 1\}
\end{cases}
\]

Proposition 2. For \( M = 2 \) codewords and any blocklength \( n \), on a BSC, the flip codes of type \( t \) for any \( t \in \{0, 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \} \) are all optimal, and on a Z-channel, the flip code of type \( 0 \) is optimal.

Theorem 3. For \( M = 3 \) or \( M = 4 \) codewords and any blocklength \( n \), on a Z-channel, an optimal choice for the codebook matrix is

\[
\mathcal{C}^{(M)} = \begin{pmatrix} c_1^{(M)} & c_2^{(M)} & c_3^{(M)} & c_4^{(M)} & \cdots \end{pmatrix}
\]

On a BSC, an optimal code for \( M = 3 \) is the weakly flip code constructed recursively by the following choice of candidate columns:

\[
\mathcal{C}^{(3)} = \begin{pmatrix} c_1^{(3)} & c_2^{(3)} & c_3^{(3)} & \cdots \end{pmatrix}
\]

and an optimal linear code for \( M = 4 \) is the weakly flip code constructed recursively by the following choice of candidate columns:

\[
\mathcal{C}^{(4)} = \begin{pmatrix} c_1^{(4)} & c_2^{(4)} & c_3^{(4)} & c_4^{(4)} & \cdots \end{pmatrix}
\]

*This work has been partially supported by the Industrial Technology Research Institute under Contract G1-98006, by the MediaTek research center at National Chiao Tung University, and by the National Science Council under NSC 100-2221-E-009-068-MY3.