Some Expectations of a Non-Central Chi-Square Distribution With an Even Number of Degrees of Freedom

Stefan M. Moser
Department of Communication Engineering
National Chiao Tung University (NCTU)
Hsinchu, Taiwan

***Abstract*—The non-central chi-square distribution plays an important role in communications, for example in the analysis of mobile and wireless communication systems. It not only includes the important cases of a squared Rayleigh distribution and a squared Rice distribution, but also the generalizations to a sum of independent squared Gaussian random variables of identical variance with or without mean, *i.e.*, a "squared MIMO Rayleigh" and "squared MIMO Rice" distribution.**

In this paper closed-form expressions are derived for the expectation of the logarithm and for the expectation of the n-th power of the reciprocal value of a non-central chi-square random variable. It is shown that these expectations can be expressed by a family of continuous functions $g_m(\cdot)$ and that these families have nice properties (monotonicity, convexity, *etc.*). Moreover, some tight upper and lower bounds are derived that are helpful in situations where the closed-form expression of $g_m(\cdot)$ is too complex for further analysis.

I. Introduction

It is well known that adding several independent squared Gaussian random variables of identical variance yields a random variable that is non-central chi-square distributed. This distribution often shows up in information theory and communications. As an example we mention the situation of a non-coherent multiple-input multiple-output (MIMO) fading channel

$$\mathbf{Y} = \mathbb{H}\mathbf{x} + \mathbf{Z} \tag{1}$$

with an additive white Gaussian noise vector $\mathbf{Z}$ and with a fading matrix $\mathbb{H}$ that consists of independent and unit-variance Gaussian distributed components with or without mean. Here conditional on the input $\mathbf{x}$ the squared magnitude of the output $\|\mathbf{Y}\|^2$ is non-central chi-square distributed.

While the special case of a squared Rayleigh distribution[1] is well understood in the sense that there exist closed-form expressions for more or less all interesting expected values, the more general situation of a non-central chi-square distribution is far more complex. Here, standard integration tools (*e.g.*, [1]) or integration lookup tables (*e.g.*, [2]) will very quickly cease to provide closed-form expressions.

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[1]A squared Rayleigh random variable is actually exponentially distributed.

In this paper we will state closed-form expressions for some of these situations: we will give closed-form solutions to $\mathsf{E}[\ln V]$ and $\mathsf{E}\left[\frac{1}{V^n}\right]$ for a non-central chi-square random variable V with an even number of degrees of freedom. Note that in practice we often have an even number of degrees of freedom because we usually consider *complex* Gaussian random variables consisting of *two real* Gaussian components. We will see that these expectations are all related to a family of functions $g_m(\cdot)$ that is defined in Definition 2 in the following section. There we will also state the main results. In Section III we will then derive some properties of the functions $g_m(\cdot)$ and in Section IV we will state tight upper and lower bounds. In Section V another property of $g_m(\cdot)$ is derived as an example of how the bounds from Section IV could be applied. We conclude in Section VI. Due to space limitations we cannot provide the complete proofs. For the proofs and more details we refer to [3], [4].

II. Definitions and Main Results

A non-negative real random variable is said to have a *non-central chi-square* distribution with n *degrees of freedom* and *non-centrality parameter* s^2 if it is distributed like

$$\sum_{j=1}^{n} (X_j + \mu_j)^2, \tag{2}$$

where $\{X_j\}_{j=1}^n$ are IID $\sim \mathcal{N}_{\mathbb{R}}(0,1)$ and the real constants $\{\mu_j\}_{j=1}^n$ satisfy

$$s^2 = \sum_{j=1}^{n} \mu_j^2. \tag{3}$$

(The distribution of (2) depends on the constants $\{\mu_j\}$ only via the sum of their squares.) The probability density function of such a distribution is given by [5, Chapter 29]

$$\frac{1}{2}\left(\frac{x}{s^2}\right)^{\frac{n-2}{4}} e^{-\frac{s^2+x}{2}} I_{n/2-1}\left(s\sqrt{x}\right), \qquad x \geq 0. \tag{4}$$

Here $I_\nu(\cdot)$ denotes the modified Bessel function of the first kind of order $\nu \in \mathbb{R}$, *i.e.*,

$$I_\nu(x) \triangleq \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{\nu+2k}, \qquad x \geq 0 \tag{5}$$

(see [2, Eq. 8.445]).

If the number of degrees of freedom n is even, *i.e.*, if $n = 2m$ for some positive integer m, then the non-central chi-square distribution can also be expressed as a sum of the squared norms of *complex* Gaussian random variables.

Definition 1: Let the random variable V have a non-central chi-square distribution with an even number $2m$ of degrees of freedom, *i.e.*,

$$V \triangleq \sum_{j=1}^{m} \left|U_j + \mu_j\right|^2 \tag{6}$$

where $\{U_j\}_{j=1}^m$ are IID $\sim \mathcal{N}_{\mathbb{C}}(0,1)$, and $\{\mu_j\}_{j=1}^m$ are complex constants. Let further the non-centrality parameter s^2 be defined as

$$s^2 \triangleq \sum_{j=1}^{m} |\mu_j|^2. \tag{7}$$

Next we define the following continuous functions.

Definition 2: The functions $g_m(\cdot)$ are defined as follows:

$$g_m(\xi) \triangleq \begin{cases} \ln(\xi) - \mathrm{Ei}(-\xi) + \sum_{j=1}^{m-1} (-1)^j \Big[e^{-\xi}(j-1)! \\ \qquad - \frac{(m-1)!}{j(m-1-j)!}\Big] \left(\frac{1}{\xi}\right)^j, & \xi > 0 \\ \psi(m), & \xi = 0 \end{cases} \tag{8}$$

for $m \in \mathbb{N}$, where $\mathrm{Ei}(\cdot)$ denotes the exponential integral function defined as

$$\mathrm{Ei}(-\xi) \triangleq -\int_{\xi}^{\infty} \frac{e^{-t}}{t}\,\mathrm{d}t, \qquad \xi > 0 \tag{9}$$

and $\psi(\cdot)$ is Euler's psi function given by

$$\psi(m) \triangleq -\gamma + \sum_{j=1}^{m-1} \frac{1}{j}, \qquad m \in \mathbb{N} \tag{10}$$

with $\gamma \approx 0.577$ denoting Euler's constant.
Note that $g_m(\xi)$ is continuous for all $\xi \geq 0$, *i.e.*, in particular

$$\lim_{\xi \downarrow 0} \Bigg\{ \ln(\xi) - \mathrm{Ei}(-\xi) + \sum_{j=1}^{m-1} (-1)^j \Big[e^{-\xi}(j-1)! - \frac{(m-1)!}{j(m-1-j)!}\Big] \left(\frac{1}{\xi}\right)^j \Bigg\} = \psi(m) \tag{11}$$

for all $m \in \mathbb{N}$. Therefore its first derivative is defined for all $\xi \geq 0$ and can be evaluated to

$$g_m'(\xi) \triangleq \frac{\partial g_m(\xi)}{\partial \xi} = \frac{(-1)^m \Gamma(m)}{\xi^m} \left(e^{-\xi} - \sum_{j=0}^{m-1} \frac{(-1)^j}{j!}\xi^j \right) \tag{12}$$

(see [6, Eq. (417)], [7, Eq. (A.39)]). Note that $g_m'(\cdot)$ is also continuous, *i.e.*, in particular

$$\lim_{\xi \downarrow 0} \left\{ \frac{(-1)^m \Gamma(m)}{\xi^m} \left(e^{-\xi} - \sum_{j=0}^{m-1} \frac{(-1)^j}{j!}\xi^j \right) \right\} = \frac{1}{m} = g_m'(0). \tag{13}$$

Now we will give closed-form expressions for some expectations of a non-central chi-square random variable. We start with the logarithm.

Theorem 3: The expected value of the logarithm of a non-central chi-square random variable with an even number $2m$ of degrees of freedom is given as

$$\boxed{\mathsf{E}[\ln V] = g_m(s^2)} \tag{14}$$

where V and s^2 are defined in (6) and (7). Hence, we have found the solution to the following integral:

$$\int_0^\infty \ln v \cdot \left(\frac{v}{s^2}\right)^{\frac{m-1}{2}} e^{-v-s^2} I_{m-1}(2s\sqrt{v})\,\mathrm{d}v = g_m(s^2) \tag{15}$$

for any $m \in \mathbb{N}$ and $s^2 \geq 0$.

Proof: A proof can be found in [6, Lemma 10.1], [7, Lemma A.6] □

Next we look at the reciprocal value.

Theorem 4: Let $n \in \mathbb{N}$ with $n < m$. The expected value of the n-th power reciprocal value of a non-central chi-square random variable with an even number $2m$ of degrees of freedom is given as

$$\boxed{\mathsf{E}\left[\frac{1}{V^n}\right] = \frac{(-1)^{n-1}}{(n-1!)} \cdot g_{m-n}^{(n)}(s^2), \qquad m > n} \tag{16}$$

where

$$g_m^{(\ell)}(\xi) = \frac{\partial^\ell g_m(\xi)}{\partial \xi^\ell} \tag{17}$$

denotes the ℓ-th derivative of $g_m(\cdot)$ and where V and s^2 are defined in (6) and (7). In particular, for $m > 1$

$$\boxed{\mathsf{E}\left[\frac{1}{V}\right] = g_{m-1}'(s^2).} \tag{18}$$

Hence, we have found the solution to the following integral:

$$\int_0^\infty \frac{1}{v^n} \cdot \left(\frac{v}{s^2}\right)^{\frac{m-1}{2}} e^{-v-s^2} I_{m-1}(2s\sqrt{v})\,\mathrm{d}v = \frac{(-1)^{n-1}}{(n-1!)} \cdot g_{m-n}^{(n)}(s^2) \tag{19}$$

for any $m, n \in \mathbb{N}$, $m > n$, and any real $s^2 \geq 0$.
Note that in the cases where $m \leq n$ the expectation is unbounded.

Proof: The proof is based on a series expansion of the modified Bessel function similarly to the proof of [6, Lemma 10.1], [7, Lemma A.6]. For more details see [3]. □

III. Properties of $g_m(\cdot)$ and $g'_m(\cdot)$

In this section we will show that the family of functions $g_m(\cdot)$ and $g'_m(\cdot)$ are well-behaved.

Corollary 5: The functions $g_m(\cdot)$ are monotonically strictly increasing and strictly concave in the interval $[0, \infty)$ for all $m \in \mathbb{N}$.

Proof: From [6, Eqs. (415), (416)], [7, Eqs. (A.37), (A.38)] we know that the first and second derivative of $g_m(\cdot)$ can be written as

$$g'_m(\xi) = e^{-\xi} \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{1}{k+m} \cdot \xi^k, \tag{20}$$

$$g''_m(\xi) = -e^{-\xi} \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{1}{(k+m)(k+m+1)} \cdot \xi^k, \tag{21}$$

i.e., the first derivative of $g_m(\cdot)$ is positive and the second derivative is negative. □

Corollary 6: The functions $g_m(\xi)$ are monotonically strictly increasing in m for all $\xi \geq 0$.

Proof: Fix two arbitrary natural numbers $m_1, m_2 \in \mathbb{N}$ such that $m_1 < m_2$. Choose $\mu_1 = s$, $\mu_2 = \ldots = \mu_{m_2} = 0$, with an arbitrary $s \geq 0$. Let $\{U_j\}_{j=1}^{m_2}$ be IID $\sim \mathcal{N}_{\mathbb{C}}(0,1)$. Then

$$\begin{aligned} & g_{m_2}(s^2) \\ &\quad = \mathsf{E}\left[\ln\left(\sum_{j=1}^{m_2} |U_j + \mu_j|^2\right)\right] && (22) \\ &\quad = \mathsf{E}\left[\ln\left(\sum_{j=1}^{m_1} |U_j + \mu_j|^2 + \sum_{j=m_1+1}^{m_2} |U_j + \mu_j|^2\right)\right] && (23) \\ &\quad > \mathsf{E}\left[\ln\left(\sum_{j=1}^{m_1} |U_j + \mu_j|^2\right)\right] && (24) \\ &\quad = g_{m_1}(s^2), && (25) \end{aligned}$$

where the first equality follows from (14); the subsequent equality from splitting the sum into two parts; the subsequent inequality from dropping some positive terms; and the final equality again from (14). □

Corollary 7: The functions $g'_m(\cdot)$ are positive, monotonically strictly decreasing, and strictly convex functions for all $m \in \mathbb{N}$.

Proof: The positivity and the monotonicity follow from (20) and (21). To prove convexity we compute the third derivative of $g_m(\cdot)$ from (21):

$$g'''_m(\xi) = e^{-\xi} \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{2}{(k+m)(k+m+1)(k+m+2)} \cdot \xi^k \tag{26}$$

which is positive. □

Corollary 8: The functions $g'_m(\xi)$ are monotonically strictly decreasing in m for all $\xi \geq 0$.

Proof: The proof is very similar to the proof of Corollary 6 and therefore omitted. For details we refer to [3]. □

Theorem 9: We have the following relations:

$$g_{m+1}(\xi) = g_m(\xi) + g'_m(\xi) \tag{27}$$

for all $m \in \mathbb{N}$ and all $\xi \geq 0$, and

$$g'_{m+1}(\xi) = \frac{1}{\xi} - \frac{m}{\xi} g'_m(\xi), \qquad \xi > 0, \tag{28}$$

and
$$g'_m(\xi) = \frac{1}{m} - \frac{\xi}{m} g'_{m+1}(\xi), \qquad \xi \geq 0, \tag{29}$$

for all $m \in \mathbb{N}$.

Proof: These relations can be proven by algebraic rearrangements using (20). For details we refer to [3]. □

IV. Bounds on $g_m(\cdot)$ and $g'_m(\cdot)$

In this section we derive some tight bounds on the functions $g_m(\cdot)$ and $g'_m(\cdot)$.

Theorem 10: The function $g'_m(\cdot)$ can be bounded as follows:

$$\frac{1}{\xi+m} \leq g'_m(\xi) \leq \min\left\{\frac{m+1}{m(\xi+m+1)}, \frac{1}{\xi+m-1}\right\}. \tag{30}$$

Note that for $\xi < m+1$ the first of the two upper bounds is tighter than second, while for $\xi > m+1$ the second is tighter. Moreover, the first upper bound coincides with $g_m(\xi)$ for $\xi = 0$, and the second upper bound is asymptotically tight when ξ tends to infinity.

Proof: These bounds follow directly from the properties derived in Section III. For details we refer to [3]. □

The bounds (30) are depicted in Figure 1 for the cases of $m = 1$, $m = 3$, and $m = 8$.

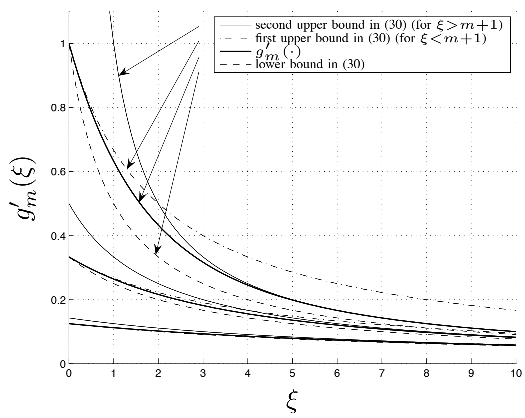

Fig. 1. Upper and lower bounds on $g'_m(\cdot)$ according to (30) in Theorem 10. The top four curves correspond to $m = 1$, the middle four to $m = 3$, and the lowest group of four curves to $m = 8$.

Theorem 11: For the functions $g_m(\cdot)$ we state two sets of bounds. The first set is tighter for smaller values of m:

$$g_m(\xi) \geq \ln \xi - \mathrm{Ei}(-\xi) + \sum_{j=1}^{m-1} \frac{1}{\xi + j}, \tag{31}$$

$$g_m(\xi) \leq \ln \xi - \mathrm{Ei}(-\xi) + \sum_{j=1}^{m-1} \min\left\{\frac{j+1}{j(\xi+j+1)}, \frac{1}{\xi+j-1}\right\}. \tag{32}$$

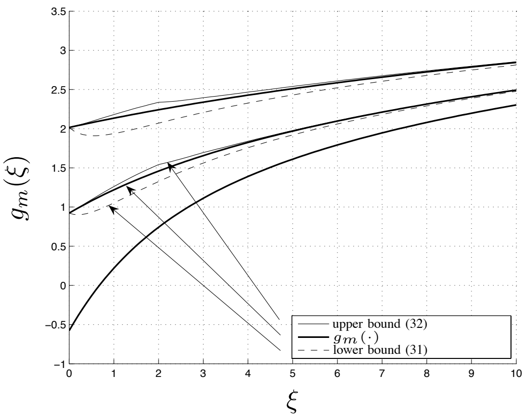

Fig. 2. Upper and lower bounds on $g_m(\cdot)$ according to (31) and (32) in Theorem 11. The lowest curve corresponds to $m = 1$ (in this case all bounds coincides with $g_1(\cdot)$), the next three curves correspond to $m = 3$, and the top three to $m = 8$.

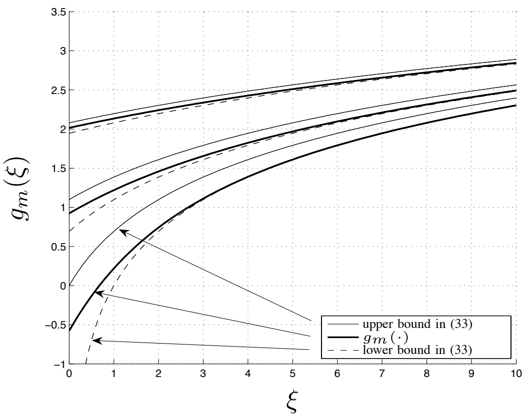

Fig. 3. Upper and lower bounds on $g_m(\cdot)$ according to (33) in Theorem 11. The lowest three curves correspond to $m = 1$, the next three to $m = 3$, and the top three to $m = 8$.

Secondly, we give a set of bounds that is tight for large values of m:

$$\ln(\xi + m - 1) \leq g_m(\xi) \leq \ln(\xi + m). \tag{33}$$

Note that this second set of bounds is very simple, *i.e.*, it is particularly interesting for further analysis.

Proof: The bounds (31) and (32) follow directly from (27) and Theorem 10. The upper bound in (33) has been proven in [8, Appendix B]. The lower bound follows from the properties derived in Section III. For details we refer to [3]. □

The bounds (31) and (32) are depicted in Figure 2 and the bounds (33) in Figure 3, both times for the cases of $m = 1$, $m = 3$, and $m = 8$.

V. Additional Properties

As an example of how the properties given in Section III and the bounds given in Section IV can be used to derive further results we state the following corollary.

Corollary 12: The functions $g_m\left(\frac{1}{\xi}\right)$ are monotonically strictly decreasing and convex in ξ for all $m \in \mathbb{N}$.

Proof: Using Theorems 9 and 10 one can easily show that

$$\frac{\partial^2}{\partial\xi^2} g_m\left(\frac{1}{\xi}\right) = g_m'\left(\frac{1}{\xi}\right) \cdot \frac{2\xi - m\xi - 1}{\xi^4} + \frac{1}{\xi^3} \tag{34}$$

$$\geq \frac{1}{\frac{1}{\xi} + m} \cdot \frac{2\xi - m\xi - 1}{\xi^4} + \frac{1}{\xi^3} \tag{35}$$

$$= \frac{2}{\xi^2(m\xi + 1)}, \tag{36}$$

which is positive and proves the convexity. □

VI. Conclusions

We have derived closed-form expressions for some important expectations in the fields of information theory and communications. We have shown that the resulting functions behave nicely and we have given tight upper and lower bounds to them.

For brevity we will not include a detailed example of how these results can be used, but only mention that the derivation of the fading number of a non-coherent MIMO Gaussian fading channel [9] strongly depends on them. There are many more examples.

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