Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Advanced Topics in Information Theory

Lecture Notes

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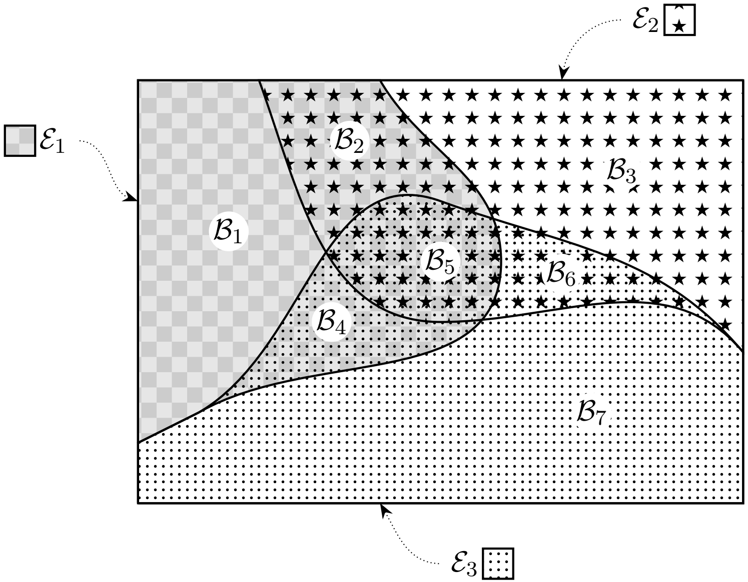
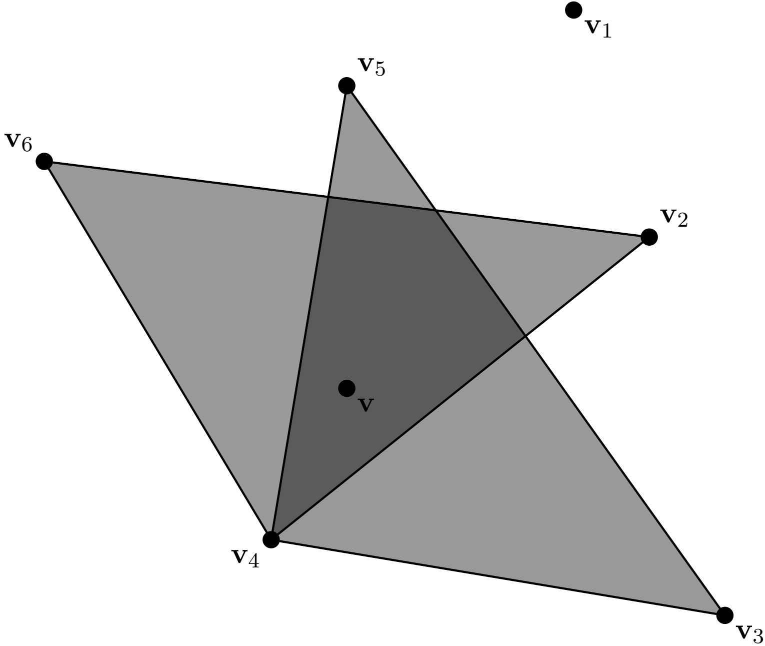
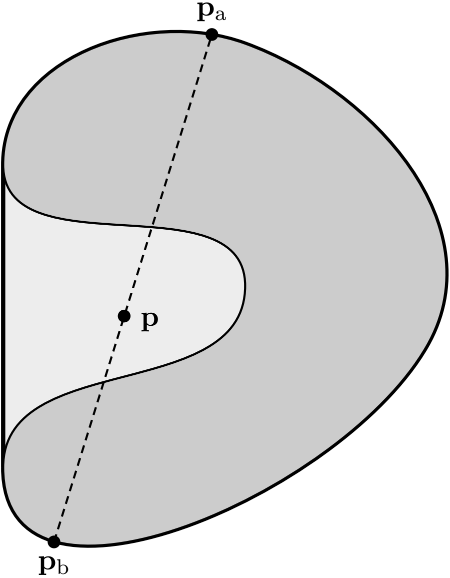
For the latest version see http://moser-isi.ethz.ch/scripts.html

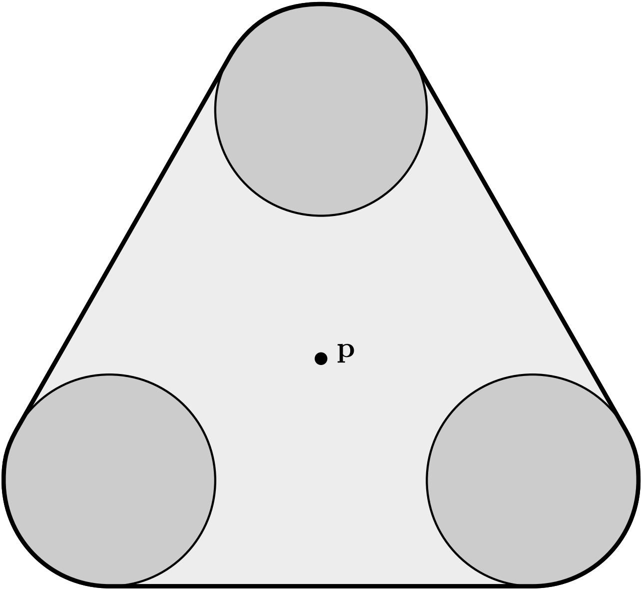


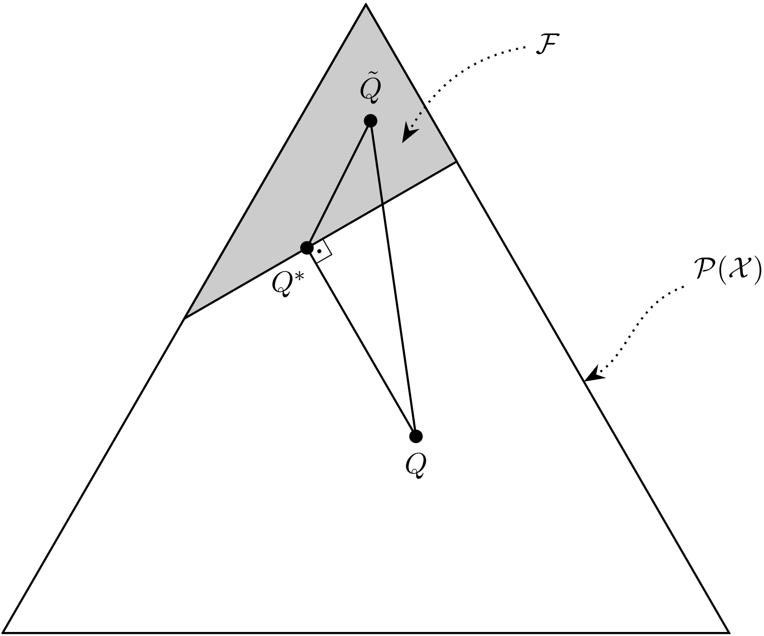
$\mathcal{E}_1$
$\mathcal{E}_2$
$\mathcal{E}_3$
$\mathcal{B}_1$
$\mathcal{B}_2$
$\mathcal{B}_3$
$\mathcal{B}_4$
$\mathcal{B}_5$
$\mathcal{B}_6$
$\mathcal{B}_7$

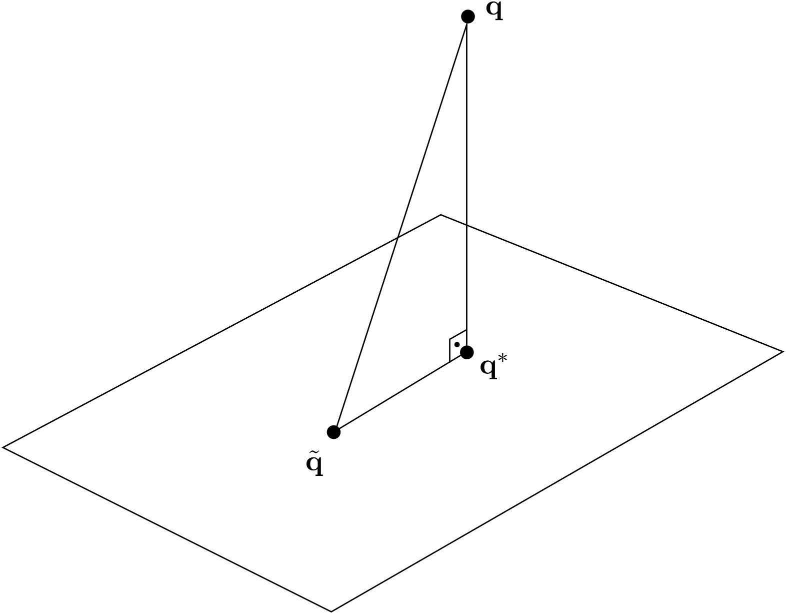

$\mathbf{v}_1$
$\mathbf{v}_5$
$\mathbf{v}_6$
$\mathbf{v}_2$
$\mathbf{v}$
$\mathbf{v}_4$
$\mathbf{v}_3$



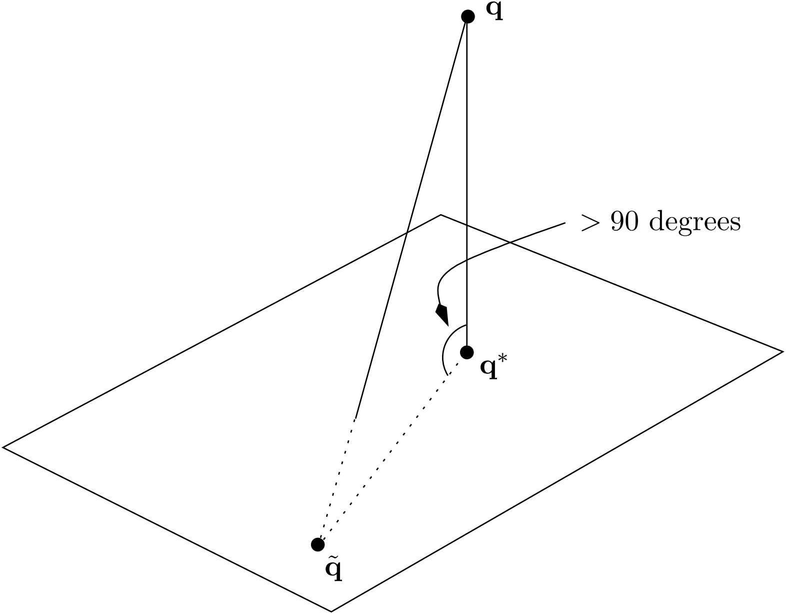
$\mathbf{p}_{\rm a}$
$\mathbf{p}$
$\mathbf{p}_{\rm b}$

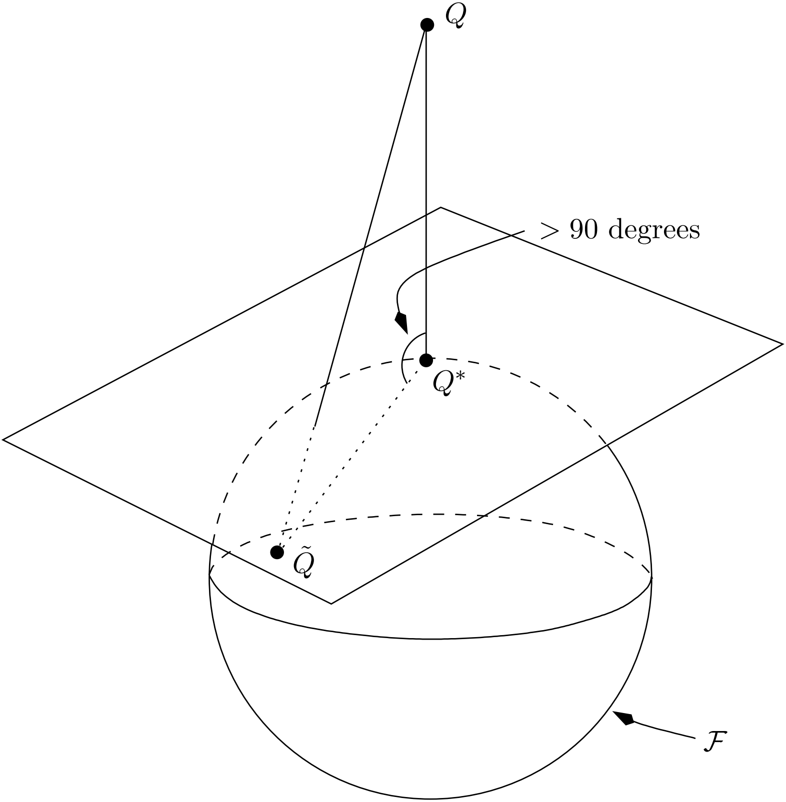

p

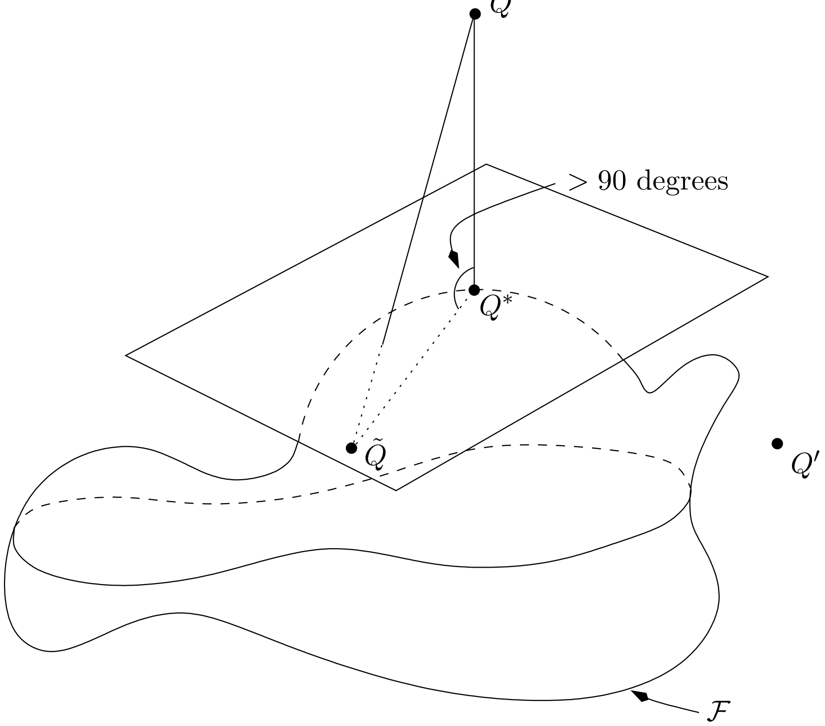

$\mathcal{F}$
$\tilde{Q}$
Q^*
Q
$\mathcal{P}(\mathcal{X})$

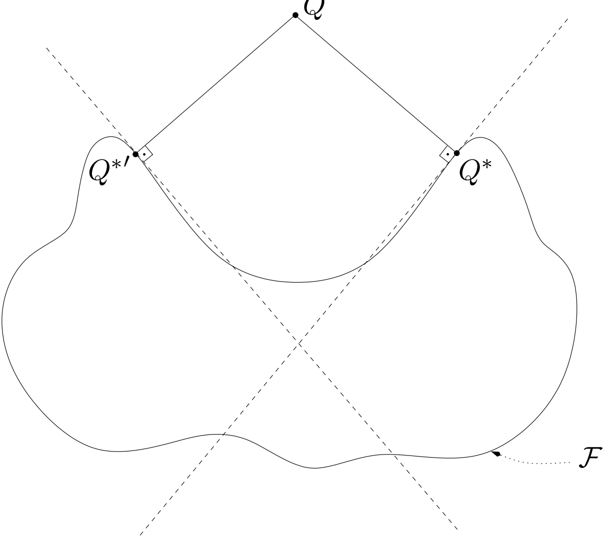


q
q*
q̃

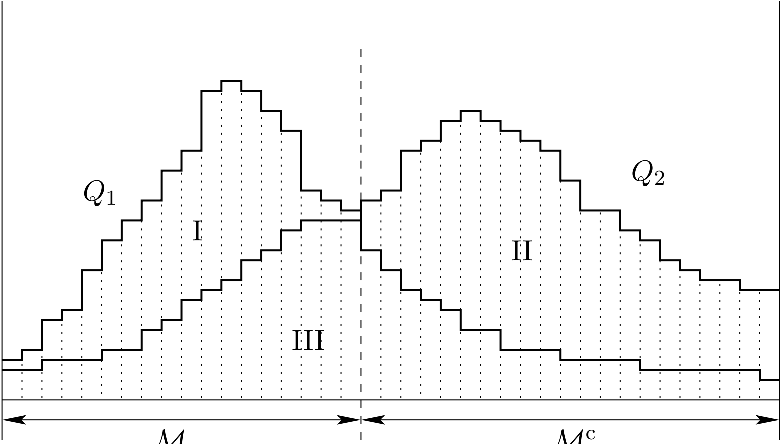

q
> 90 degrees
q*
q̃

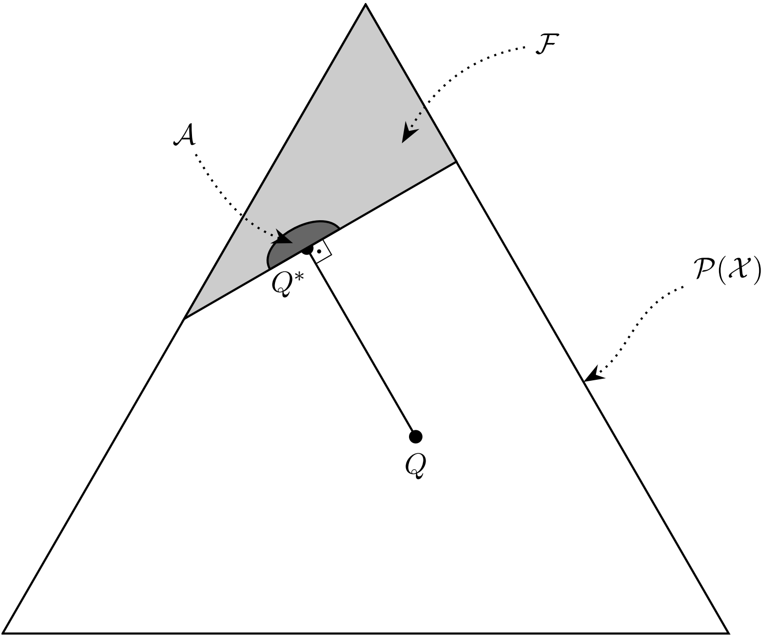

Q
> 90 degrees
Q^*
$\tilde{Q}$
$\mathcal{F}$

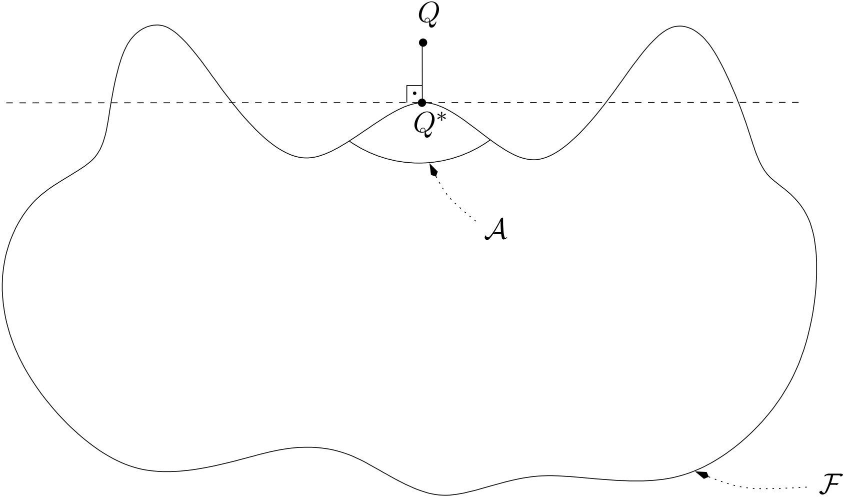

Q
> 90 degrees
Q^*
$\tilde{Q}$
Q'
$\mathcal{F}$



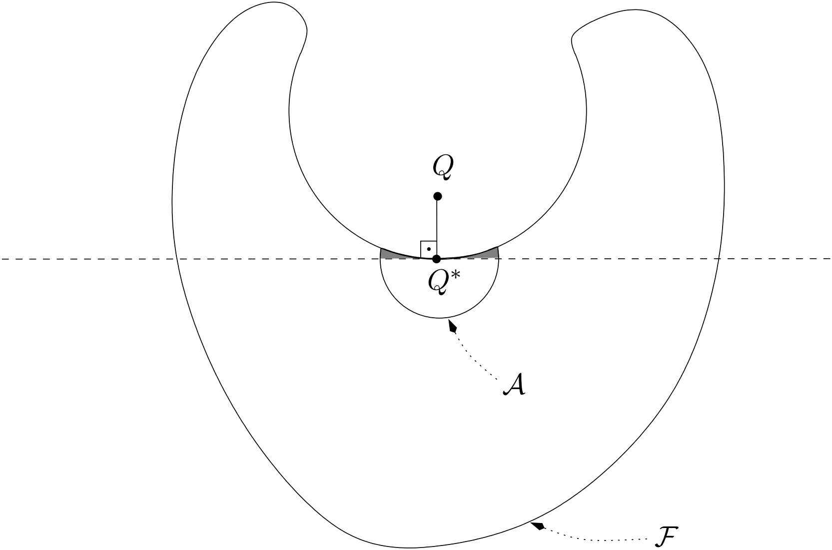
Q
$Q^{*\prime}$
Q^{*}
$\mathcal{F}$

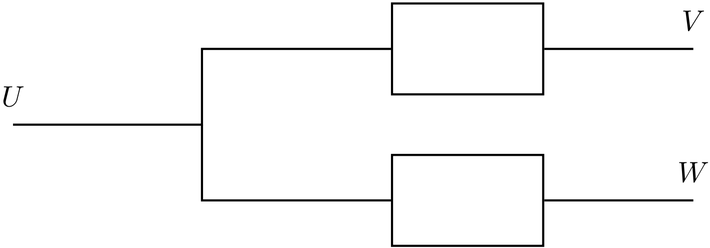

Q_1
I
Q_2
II
III
$\mathcal{M}$
$\mathcal{M}^c$


$\mathcal{F}$
$\mathcal{A}$
Q^*
Q
$\mathcal{P}(\mathcal{X})$

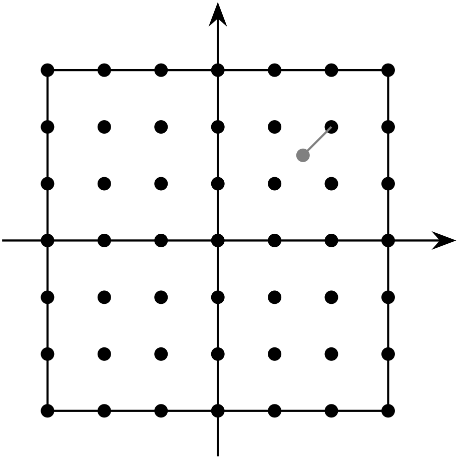


Q
Q^*
$\mathcal{A}$
$\mathcal{F}$

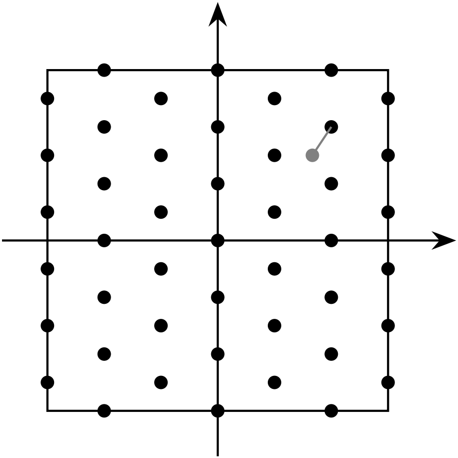

Q
Q^*
$\mathcal{A}$
$\mathcal{F}$



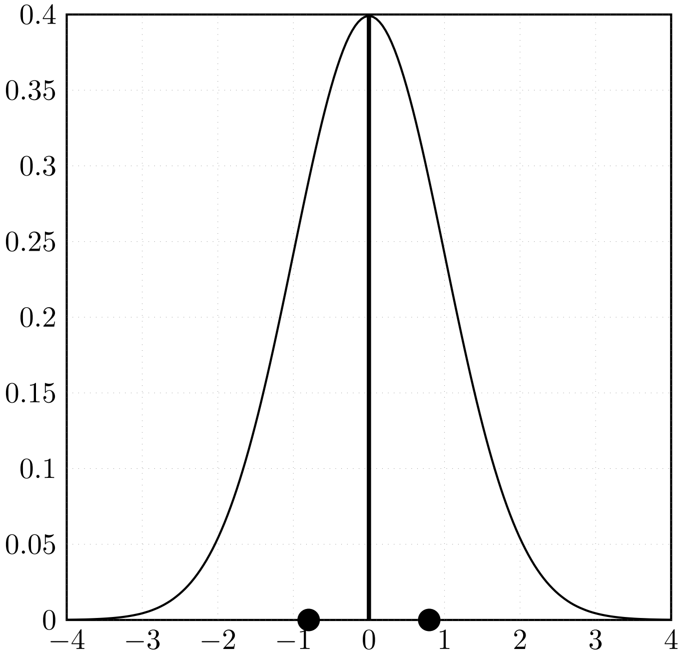
U
V
W



independent description
49 points

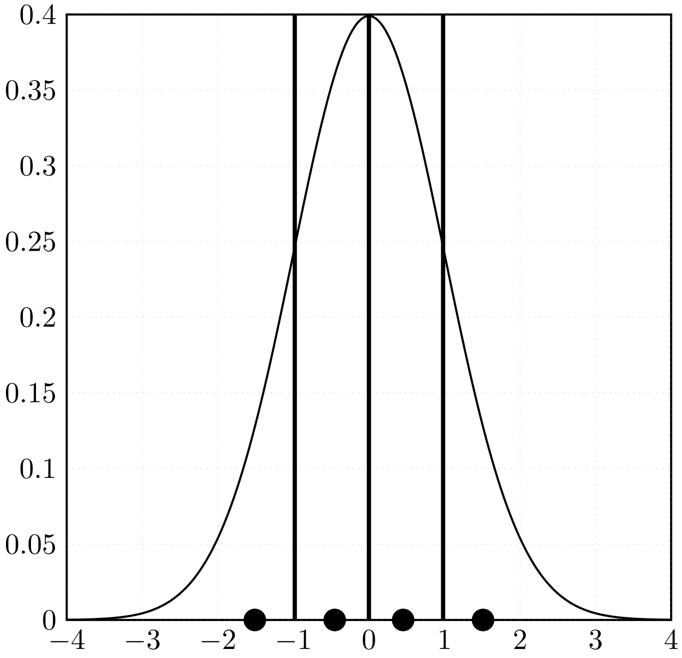


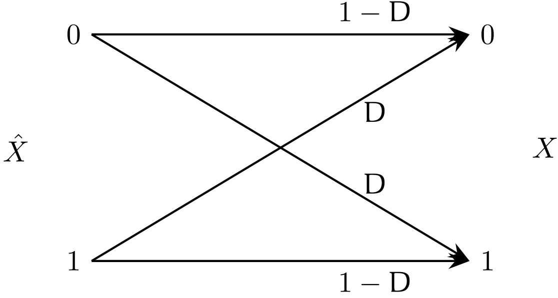
dependent description
45 points

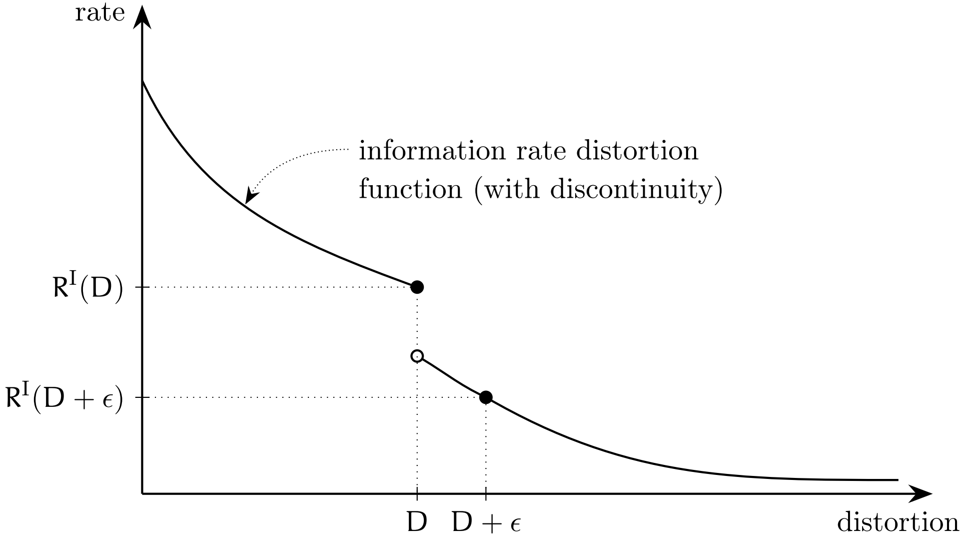


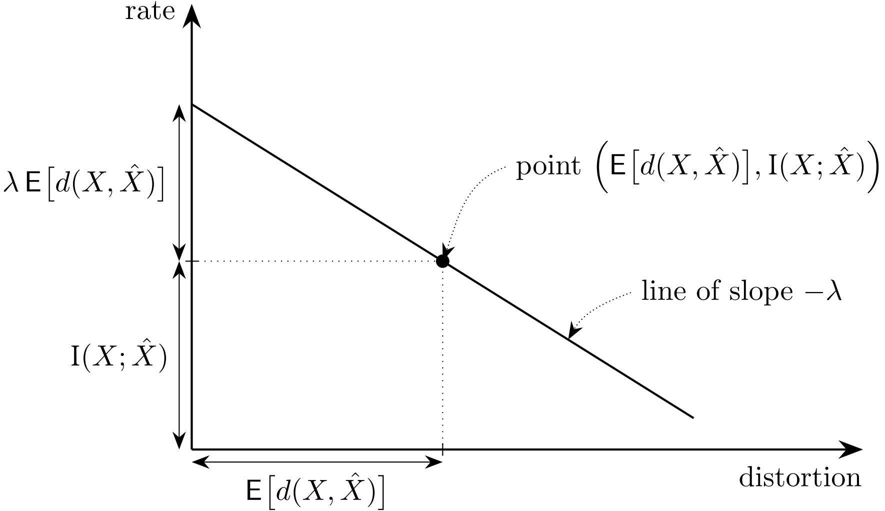
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
−4
−3
−2
−1
0
1
2
3
4

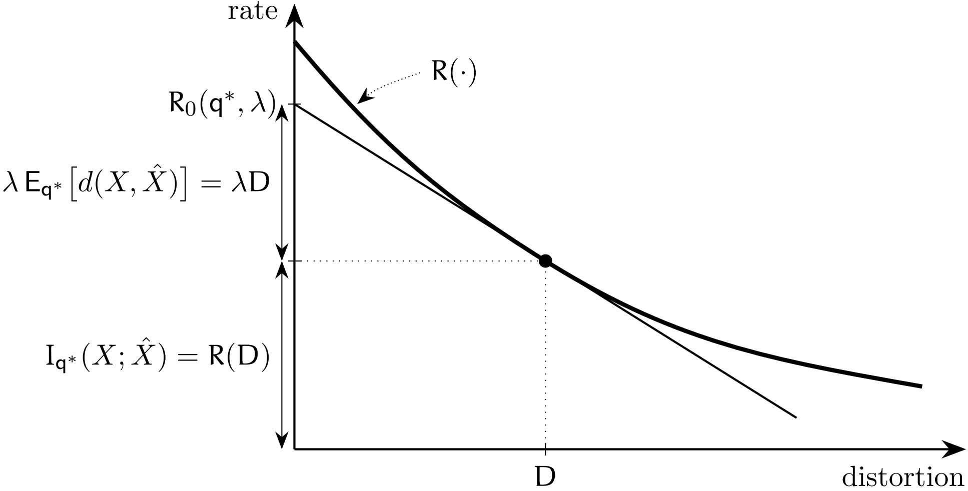
$\hat{x}_1$	$\hat{x}_2$	θ
0.5	1	0.75
0.3578	1.3288	0.8433
0.3973	1.4011	0.8992
0.4202	1.4450	0.9326
0.4336	1.4714	0.9525
0.4414	1.4872	0.9643
0.4461	1.4966	0.9714
0.4488	1.5022	0.9755
	$\vdots$	
0.4528	1.5104	0.9816

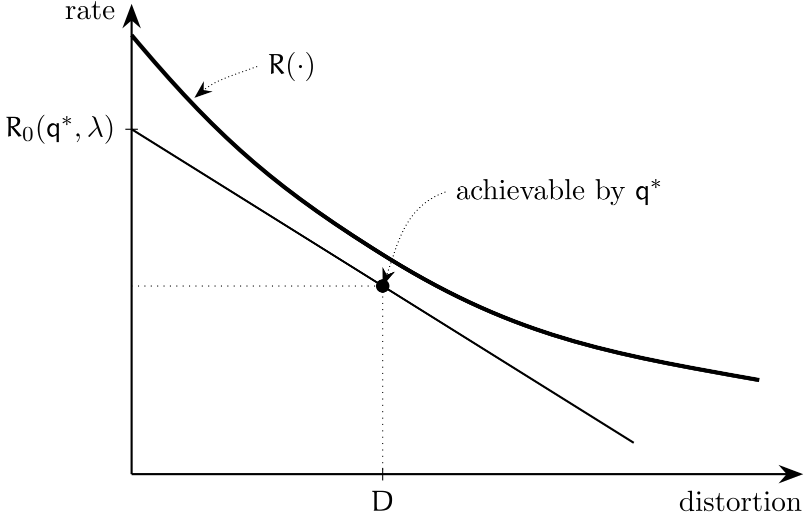

0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
−4
−3
−2
−1
0
1
2
3
4

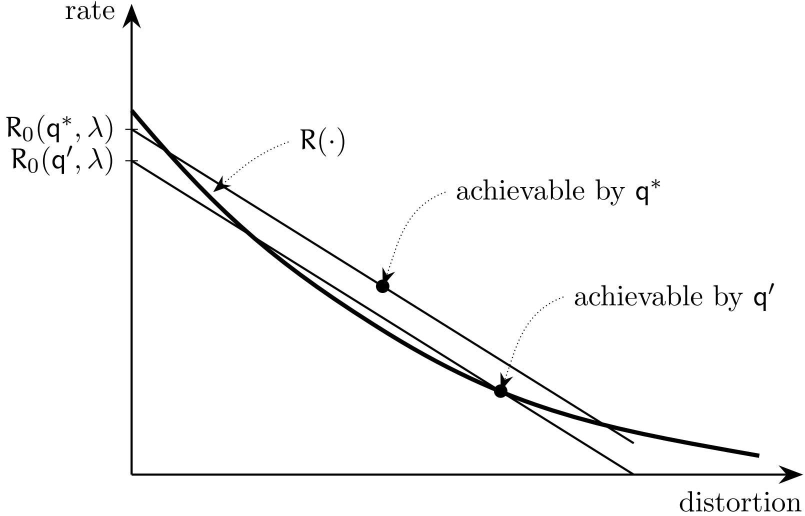

1 − D
0
0
D
$\hat{X}$
X
D
1
1
1 − D

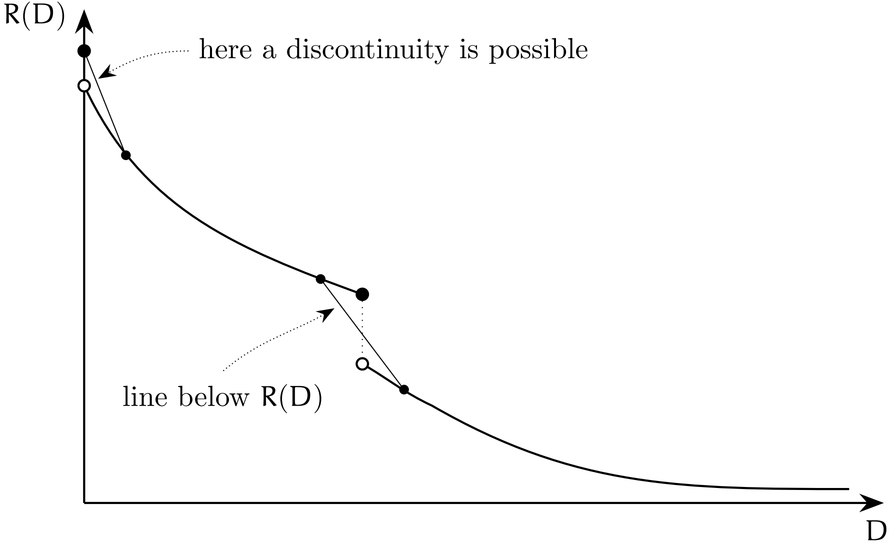

rate
information rate distortion
function (with discontinuity)
$\mathsf{R}^{\mathrm{I}}(\mathsf{D})$
$\mathsf{R}^{\mathrm{I}}(\mathsf{D}+\epsilon)$
D
$\mathsf{D}+\epsilon$
distortion

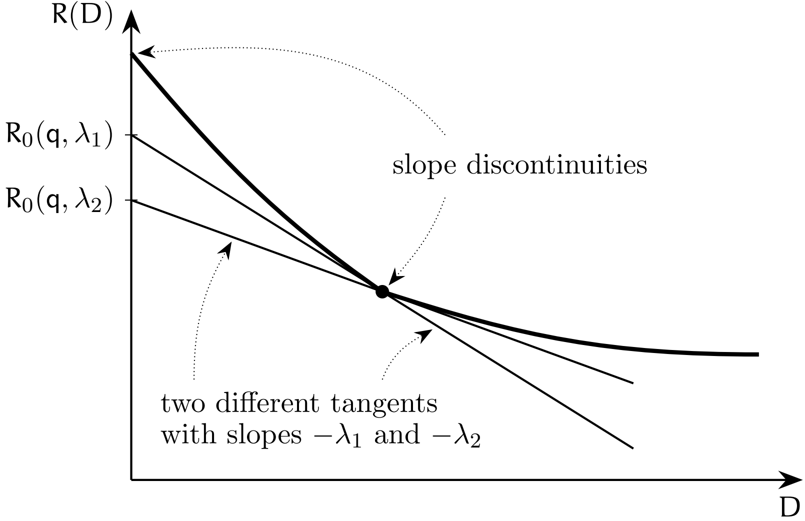

rate
$\lambda\,\mathsf{E}\big[d(X,\hat{X})\big]$
point $\Big(\mathsf{E}\big[d(X,\hat{X})\big], \mathsf{I}(X;\hat{X})\Big)$
line of slope $-\lambda$
$\mathsf{I}(X;\hat{X})$
distortion
$\mathsf{E}\big[d(X,\hat{X})\big]$

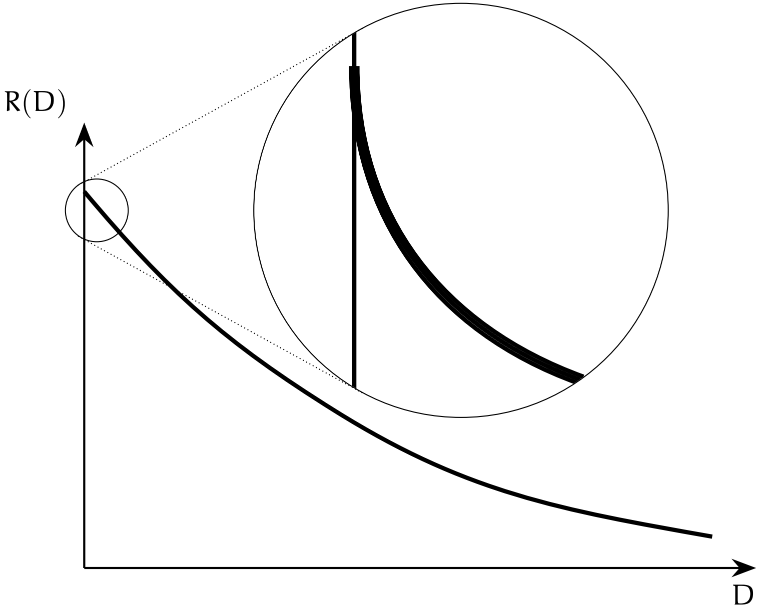

rate
$\mathsf{R}(\cdot)$
$\mathsf{R}_0(\mathsf{q}^*, \lambda)$
$\lambda\, \mathsf{E}_{\mathsf{q}^*}\big[d(X, \hat{X})\big] = \lambda \mathsf{D}$
$\mathrm{I}_{\mathsf{q}^*}(X; \hat{X}) = \mathsf{R}(\mathsf{D})$
D
distortion


rate
R(·)
R0(q*, λ)
achievable by q*
D
distortion

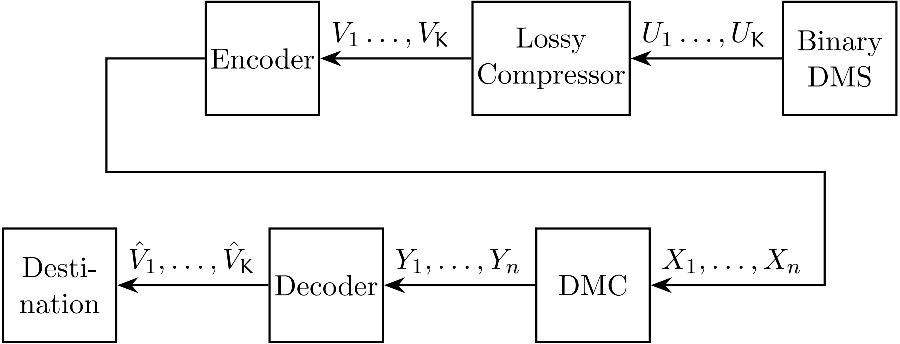

rate
$\mathsf{R}_0(\mathsf{q}^*, \lambda)$
$\mathsf{R}_0(\mathsf{q}', \lambda)$
$\mathsf{R}(\cdot)$
achievable by q^*
achievable by q'
distortion

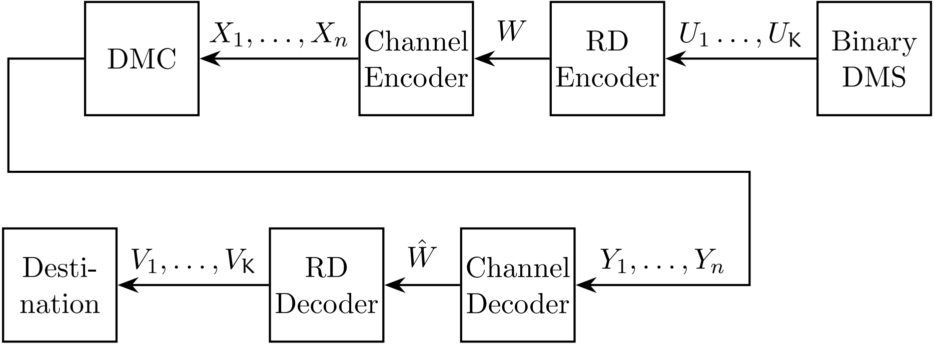

R(D)
here a discontinuity is possible
line below R(D)
D

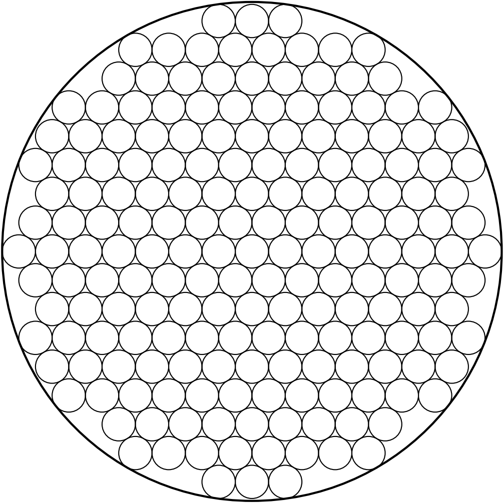

R(D)
$R_0(q, \lambda_1)$
$R_0(q, \lambda_2)$
slope discontinuities
two different tangents
with slopes $-\lambda_1$ and $-\lambda_2$
D

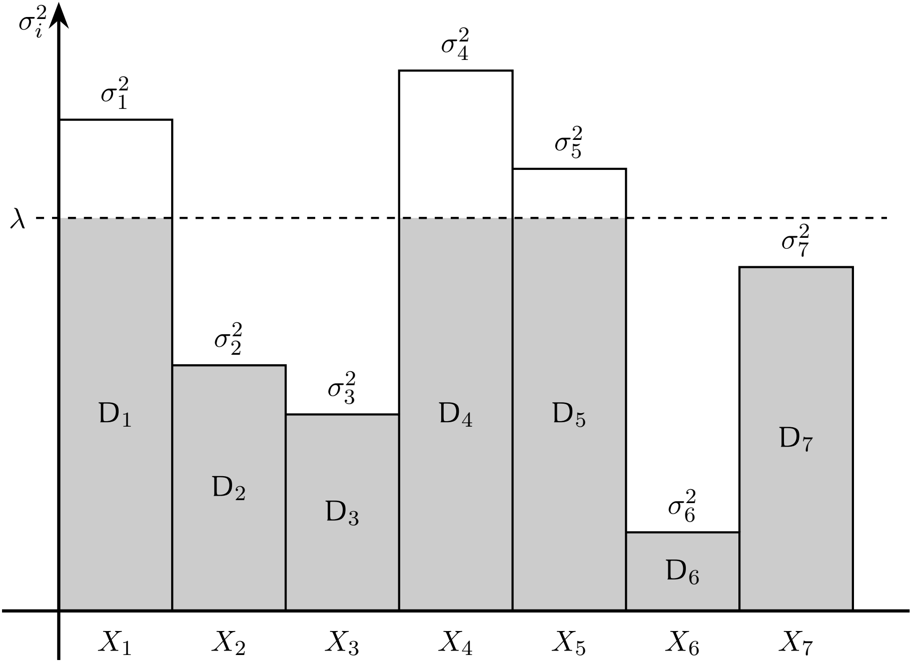

R(D)
D

Stefan M. Moser, *Advanced Topics in Information Theory*, Version 2.10.

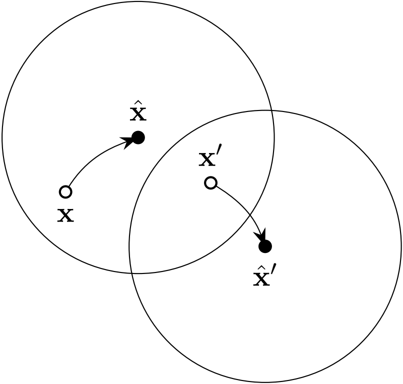

Encoder
$V_1 \ldots, V_\mathsf{K}$
Lossy Compressor
$U_1 \ldots, U_\mathsf{K}$
Binary DMS
Desti-nation
$\hat{V}_1, \ldots, \hat{V}_\mathsf{K}$
Decoder
$Y_1, \ldots, Y_n$
DMC
$X_1, \ldots, X_n$


DMC
$X_1, \ldots, X_n$
Channel Encoder
W
RD Encoder
$U_1 \ldots, U_{\mathsf{K}}$
Binary DMS
Desti-nation
$V_1, \ldots, V_{\mathsf{K}}$
RD Decoder
$\hat{W}$
Channel Decoder
$Y_1, \ldots, Y_n$

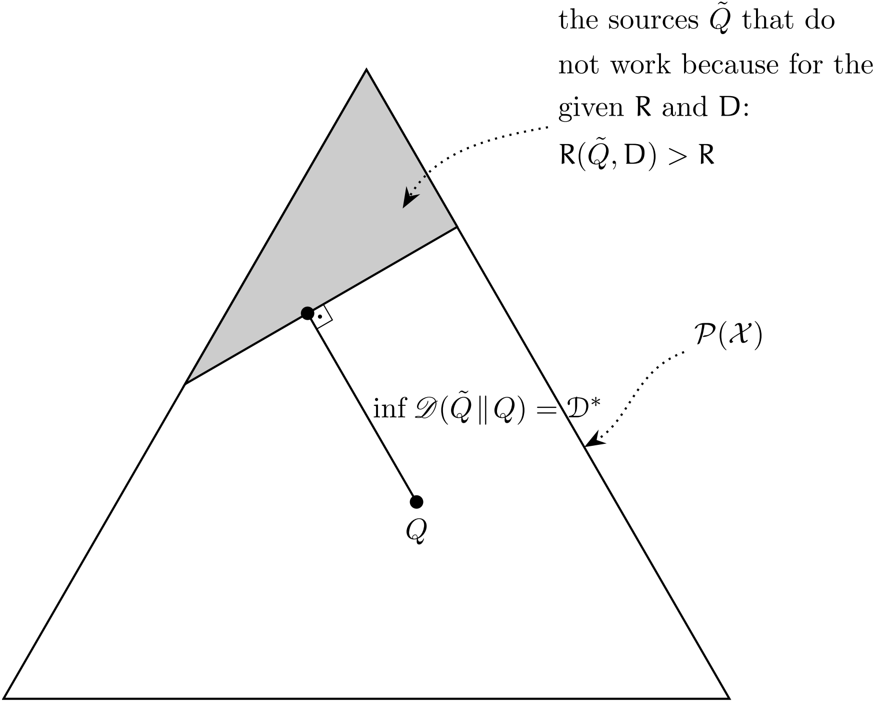




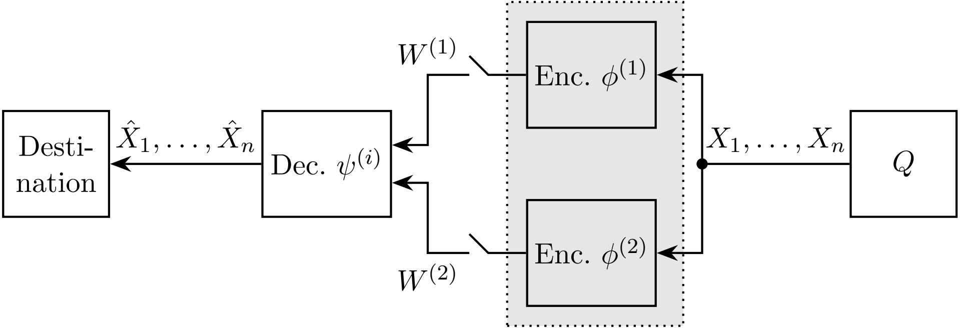
σ_i^2
σ_1^2
σ_2^2
σ_3^2
σ_4^2
σ_5^2
σ_6^2
σ_7^2
λ
D_1
D_2
D_3
D_4
D_5
D_6
D_7
X_1
X_2
X_3
X_4
X_5
X_6
X_7



$\hat{\mathbf{x}}$
$\mathbf{x}'$
$\mathbf{x}$
$\hat{\mathbf{x}}'$



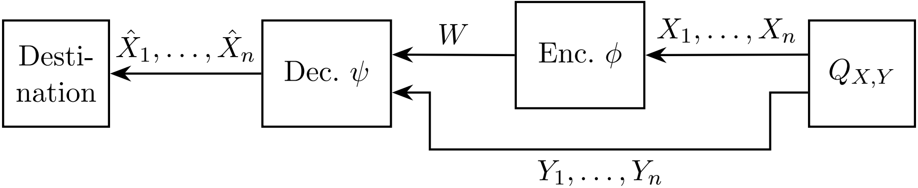
the sources $\tilde{Q}$ that do not work because for the given R and D:
$\mathsf{R}(\tilde{Q}, \mathsf{D}) > \mathsf{R}$
$\mathcal{P}(\mathcal{X})$
$\inf \mathscr{D}(\tilde{Q} \,\|\, Q) = \mathcal{D}^*$
Q

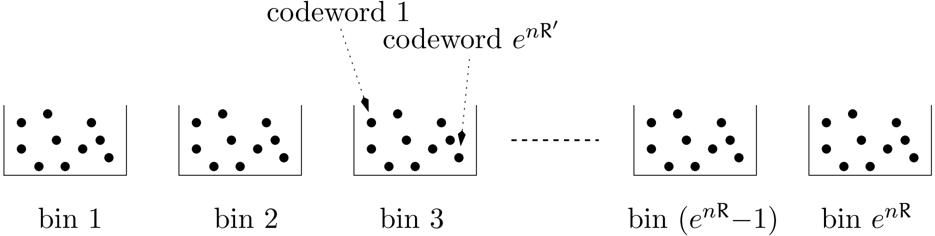

$W^{(1)}$
Enc. $\phi^{(1)}$
Destination
$\hat{X}_1, \ldots, \hat{X}_n$
Dec. $\psi^{(i)}$
$X_1, \ldots, X_n$
Q
Enc. $\phi^{(2)}$
$W^{(2)}$

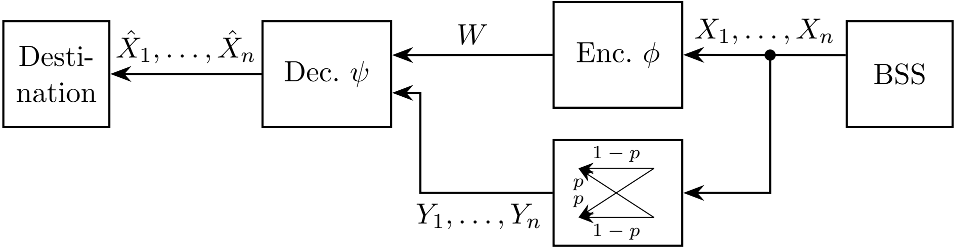
		$\hat{X}^{(2)}$		
	$Q_{\hat{X}^{(1)},\hat{X}^{(2)}\|X}(\cdot,\cdot\|0)$	0	1	$Q_{\hat{X}^{(1)}\|X}(\cdot\|0)$
$\hat{X}^{(1)}$	0	$3-2\sqrt{2}$	$\sqrt{2}-1$	$2-\sqrt{2}$
	1	$\sqrt{2}-1$	0	$\sqrt{2}-1$
	$Q_{\hat{X}^{(2)}\|X}(\cdot\|0)$	$2-\sqrt{2}$	$\sqrt{2}-1$	

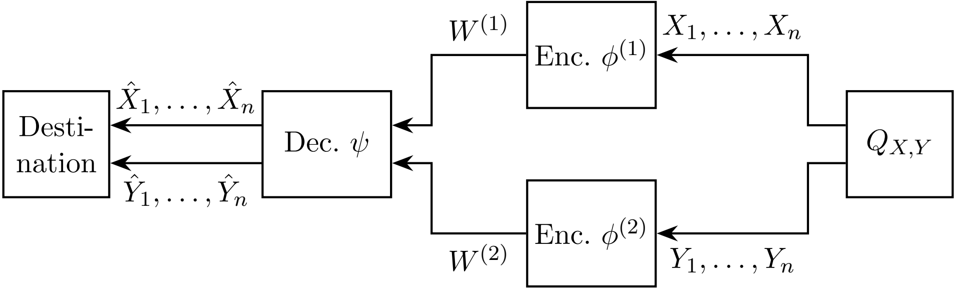
		$\hat{X}^{(2)}$		
	$Q_{\hat{X}^{(1)},\hat{X}^{(2)}\|X}(\cdot,\cdot\|1)$	0	1	$Q_{\hat{X}^{(1)}\|X}(\cdot\|1)$
$\hat{X}^{(1)}$	0	0	0	0
	1	0	1	1
	$Q_{\hat{X}^{(2)}\|X}(\cdot\|1)$	0	1	

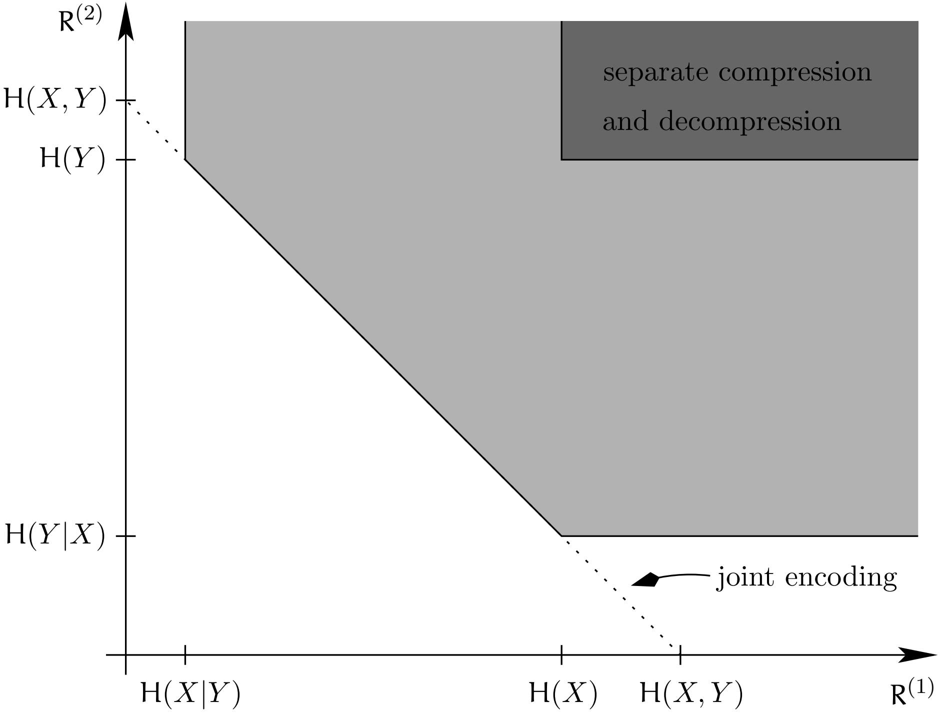
		$\hat{X}^{(2)}$		
	$Q_{\hat{X}^{(1)},\hat{X}^{(2)}}(\cdot,\cdot)$	0	1	$Q_{\hat{X}^{(1)}}(\cdot)$
$\hat{X}^{(1)}$	0	$\frac{3}{2}-\sqrt{2}$	$\frac{\sqrt{2}}{2}-\frac{1}{2}$	$1-\frac{\sqrt{2}}{2}$
	1	$\frac{\sqrt{2}}{2}-\frac{1}{2}$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$
	$Q_{\hat{X}^{(2)}}(\cdot)$	$1-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	


Destination
$\hat{X}_1, \ldots, \hat{X}_n$
Dec. ψ
W
Enc. ϕ
$X_1, \ldots, X_n$
$Q_{X,Y}$
$Y_1, \ldots, Y_n$


codeword 1
codeword $e^{n\mathsf{R}'}$
bin 1
bin 2
bin 3
bin $(e^{n\mathsf{R}}-1)$
bin $e^{n\mathsf{R}}$

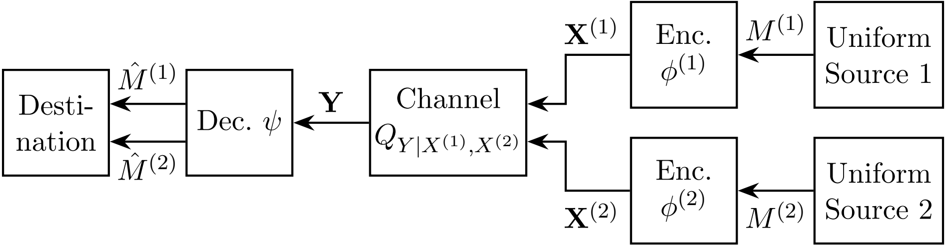

Desti-
nation
$\hat{X}_1, \ldots, \hat{X}_n$
Dec. ψ
W
Enc. ϕ
$X_1, \ldots, X_n$
BSS
$1-p$
p
p
$1-p$
$Y_1, \ldots, Y_n$

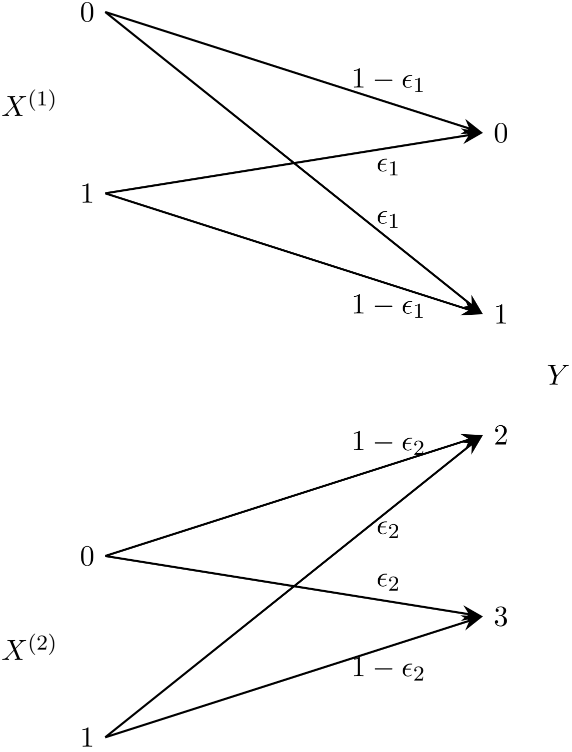

$W^{(1)}$
Enc. $\phi^{(1)}$
$X_1, \ldots, X_n$
Destination
$\hat{X}_1, \ldots, \hat{X}_n$
Dec. ψ
$Q_{X,Y}$
$\hat{Y}_1, \ldots, \hat{Y}_n$
Enc. $\phi^{(2)}$
$W^{(2)}$
$Y_1, \ldots, Y_n$



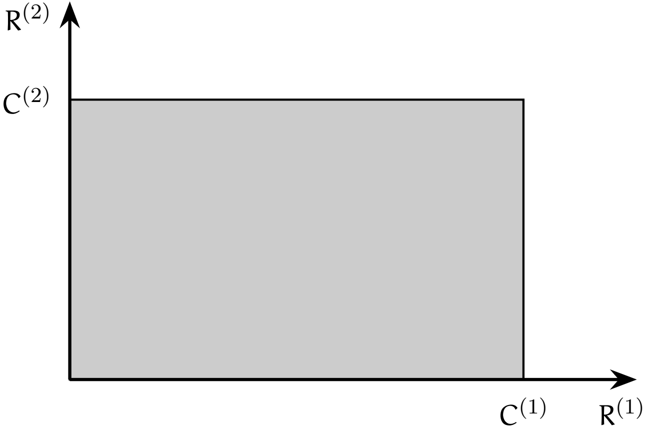
$\mathsf{R}^{(2)}$
$\mathsf{H}(X,Y)$
$\mathsf{H}(Y)$
$\mathsf{H}(Y|X)$
separate compression
and decompression
joint encoding
$\mathsf{H}(X|Y)$
$\mathsf{H}(X)$
$\mathsf{H}(X,Y)$
$\mathsf{R}^{(1)}$

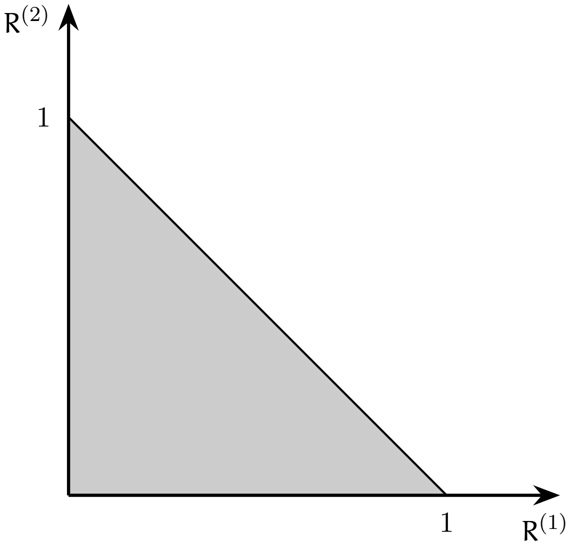
	$Q_{X,Y}(\cdot,\cdot)$	Taichung Y: rain	Taichung Y: sun	Hsinchu total
Hsinchu X	rain	0.445	0.055	0.5
	sun	0.055	0.445	0.5
	Taichung total	0.5	0.5	


Destination
$\hat{M}^{(1)}$
$\hat{M}^{(2)}$
Dec. ψ
$\mathbf{Y}$
Channel $Q_{Y|X^{(1)},X^{(2)}}$
$\mathbf{X}^{(1)}$
$\mathbf{X}^{(2)}$
Enc. $\phi^{(1)}$
Enc. $\phi^{(2)}$
$M^{(1)}$
$M^{(2)}$
Uniform Source 1
Uniform Source 2

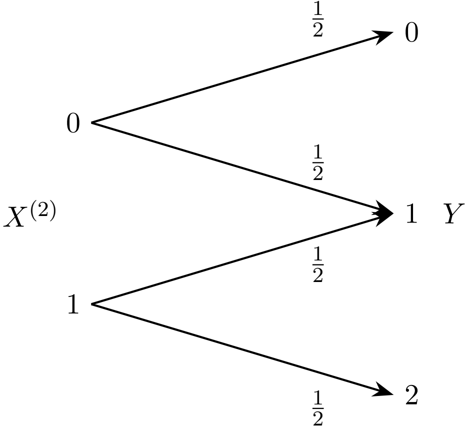


0
$X^{(1)}$
1
$1-\epsilon_1$
ϵ_1
ϵ_1
$1-\epsilon_1$
0
1
Y
2
3
$1-\epsilon_2$
ϵ_2
ϵ_2
$1-\epsilon_2$
0
$X^{(2)}$
1

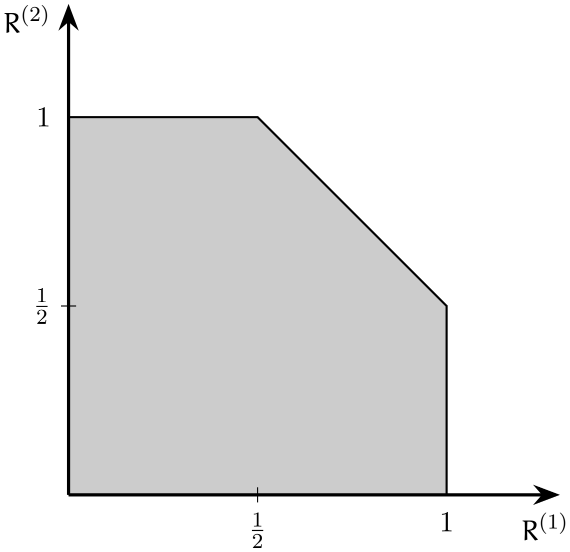

$R^{(2)}$
$C^{(2)}$
$C^{(1)}$
$R^{(1)}$



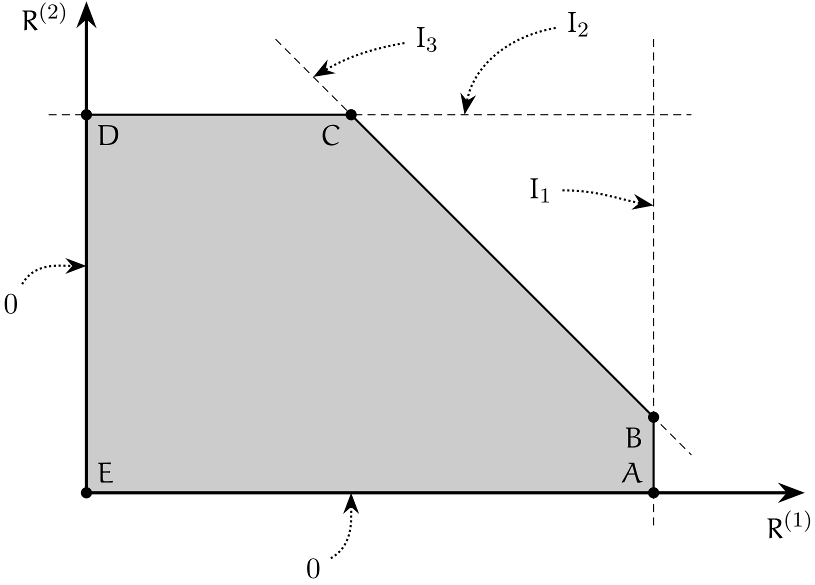
$R^{(2)}$
1
1
$R^{(1)}$



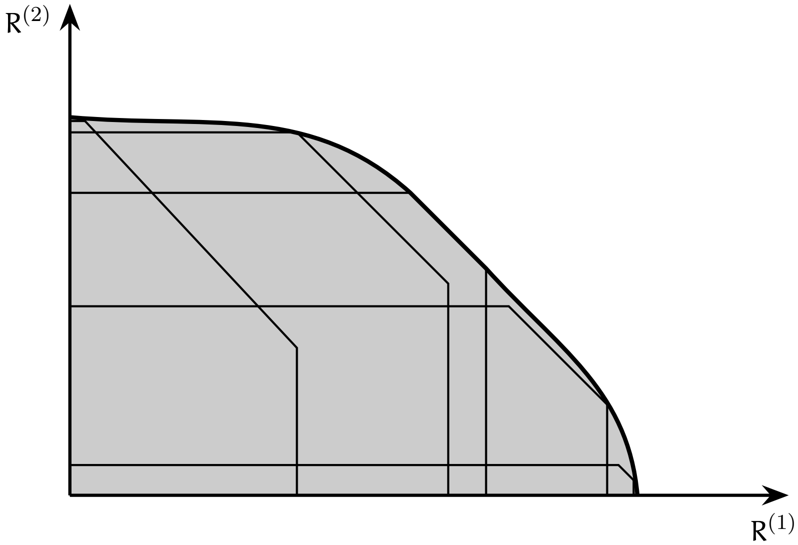
$\frac{1}{2}$
0
0
$\frac{1}{2}$
$X^{(2)}$
1
Y
$\frac{1}{2}$
1
2
$\frac{1}{2}$

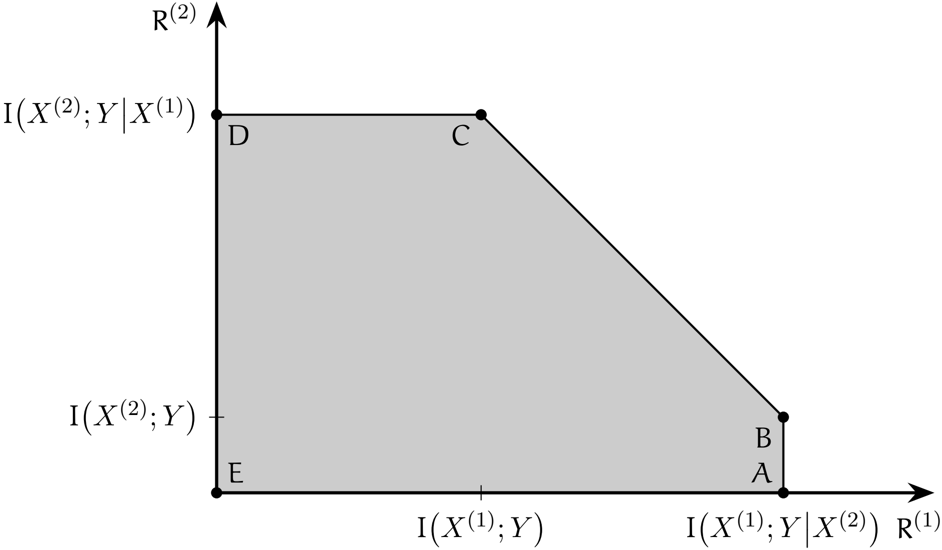


$R^{(2)}$
1
$\frac{1}{2}$
$\frac{1}{2}$
1
$R^{(1)}$

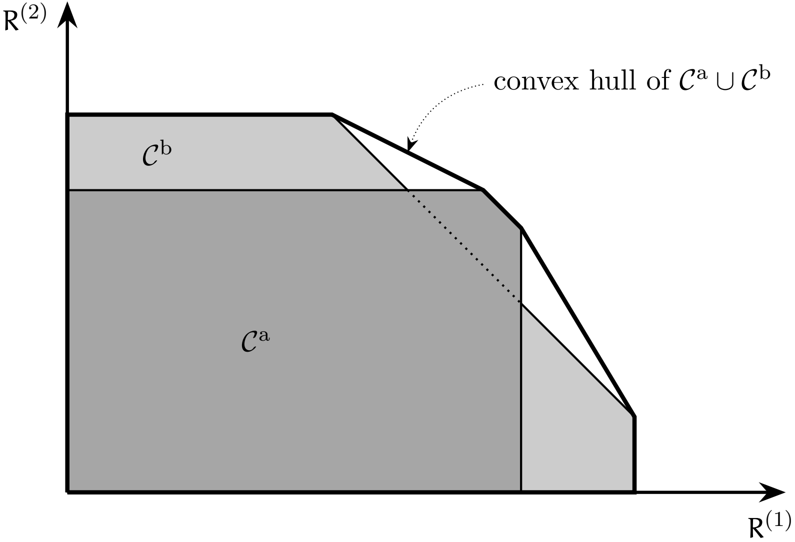


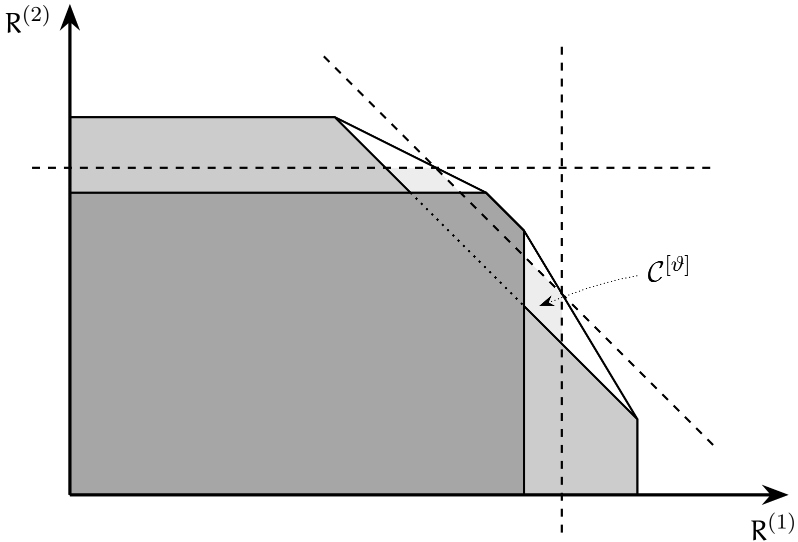
$R^{(2)}$
I_3
I_2
I_1
D
C
B
A
E
0
0
$R^{(1)}$

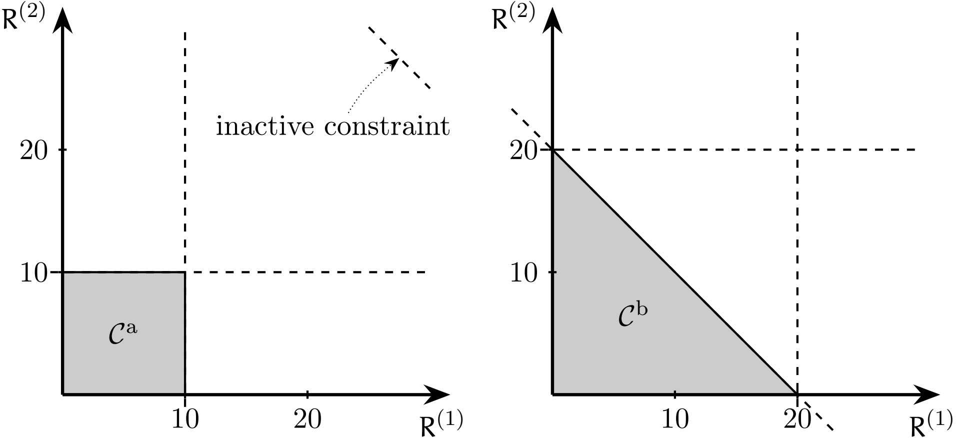

$R^{(2)}$
$R^{(1)}$



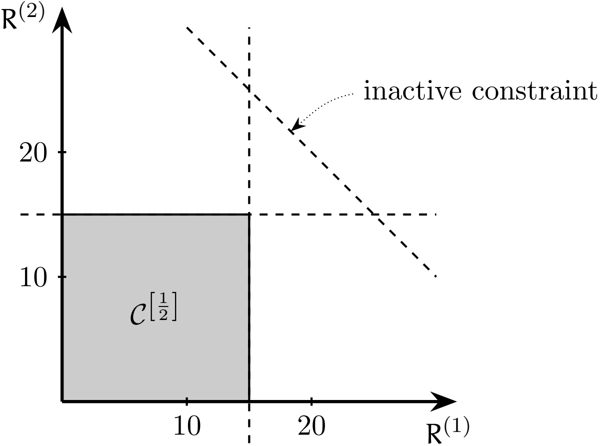
$\mathsf{R}^{(2)}$
$\mathrm{I}\big(X^{(2)};Y\big|X^{(1)}\big)$
D
C
$\mathrm{I}\big(X^{(2)};Y\big)$
B
E
A
$\mathrm{I}\big(X^{(1)};Y\big)$
$\mathrm{I}\big(X^{(1)};Y\big|X^{(2)}\big)$
$\mathsf{R}^{(1)}$

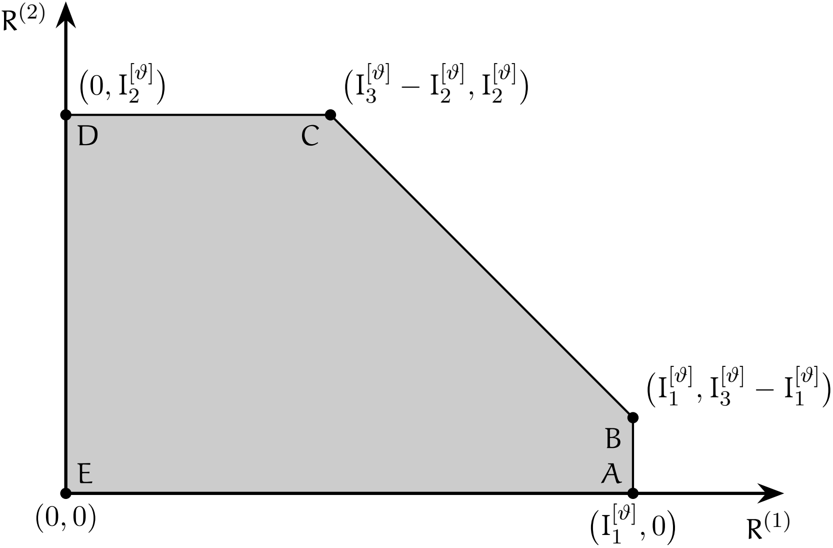

$\mathsf{R}^{(2)}$
convex hull of $\mathcal{C}^{\mathrm{a}} \cup \mathcal{C}^{\mathrm{b}}$
$\mathcal{C}^{\mathrm{b}}$
$\mathcal{C}^{\mathrm{a}}$
$\mathsf{R}^{(1)}$

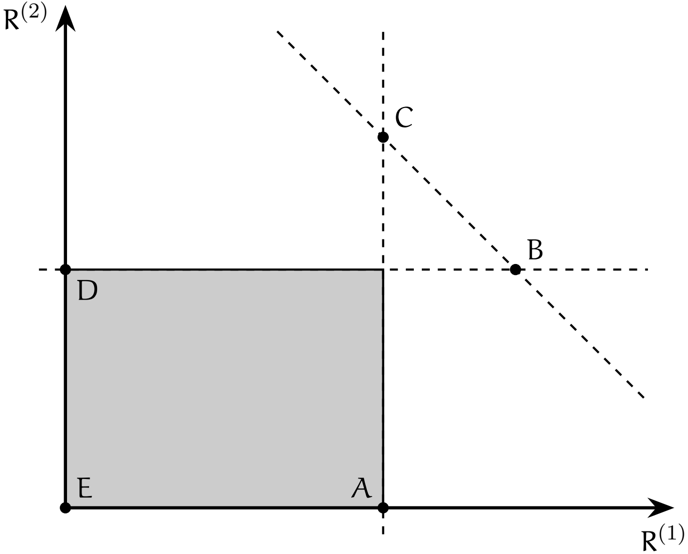

$R^{(2)}$
$\mathcal{C}^{[\vartheta]}$
$R^{(1)}$



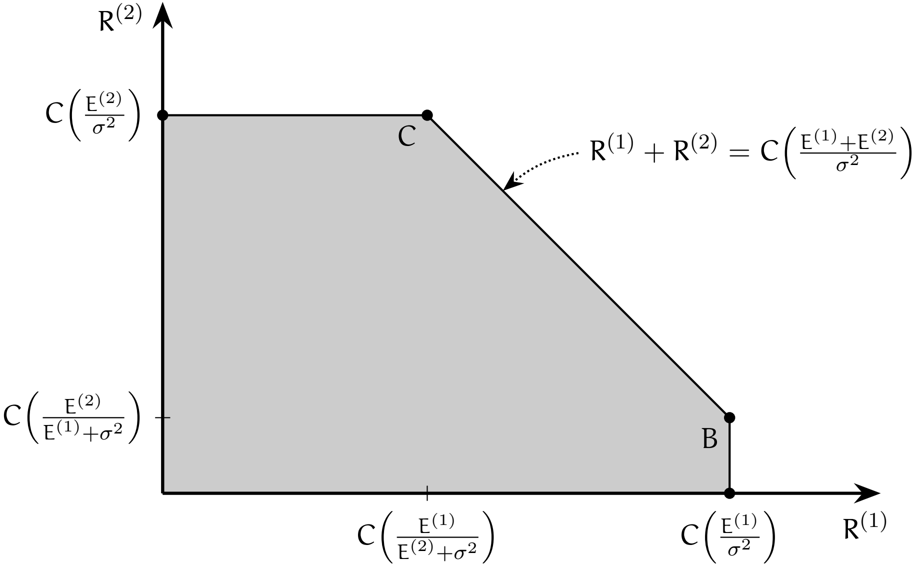
$R^{(2)}$
inactive constraint
20
10
$\mathcal{C}^{a}$
10
20
$R^{(1)}$
$R^{(2)}$
20
10
$\mathcal{C}^{b}$
10
20
$R^{(1)}$


$R^{(2)}$
inactive constraint
20
10
$\mathcal{C}^{[\frac{1}{2}]}$
10
20
$R^{(1)}$

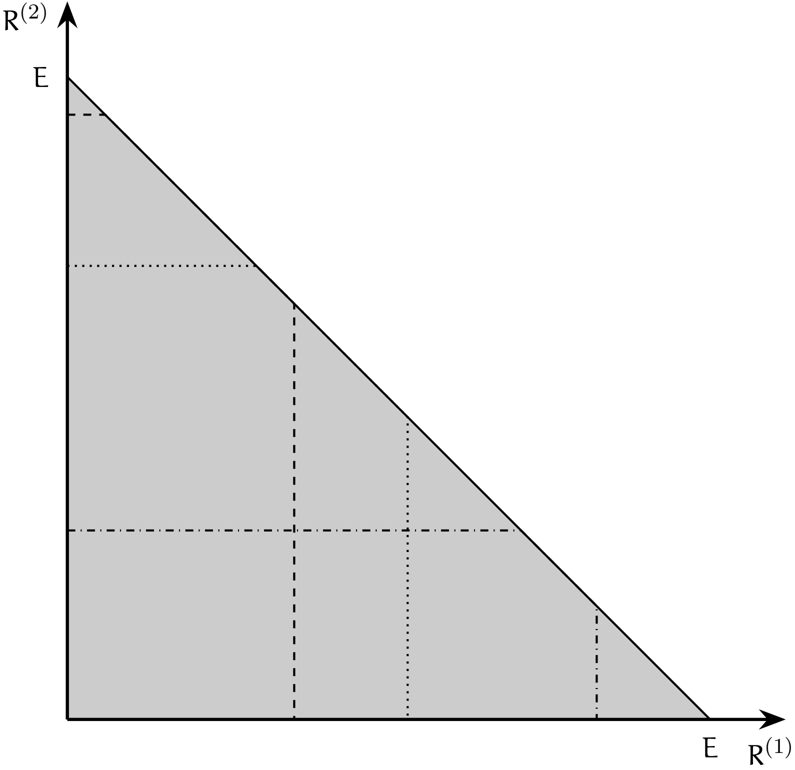

$R^{(2)}$
$\left(0, I_2^{[\vartheta]}\right)$
$\left(I_3^{[\vartheta]} - I_2^{[\vartheta]}, I_2^{[\vartheta]}\right)$
D
C
$\left(I_1^{[\vartheta]}, I_3^{[\vartheta]} - I_1^{[\vartheta]}\right)$
B
E
A
$(0, 0)$
$\left(I_1^{[\vartheta]}, 0\right)$
$R^{(1)}$



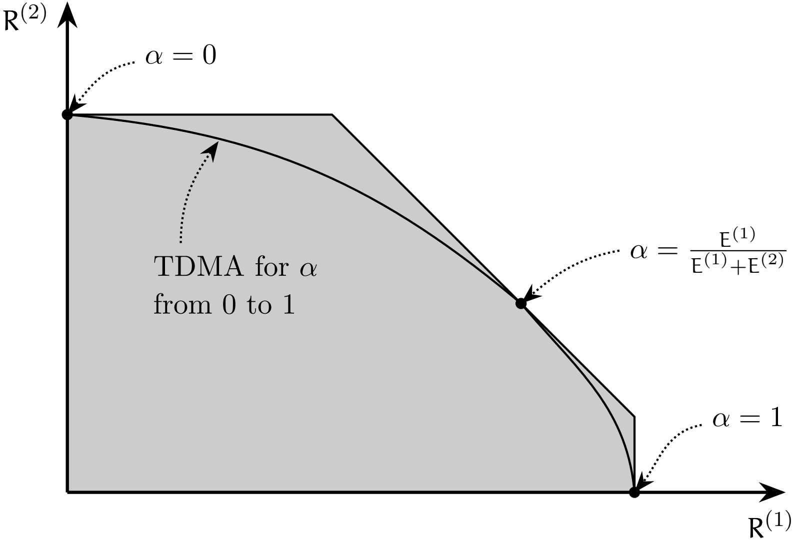
$R^{(2)}$
C
B
D
E
A
$R^{(1)}$



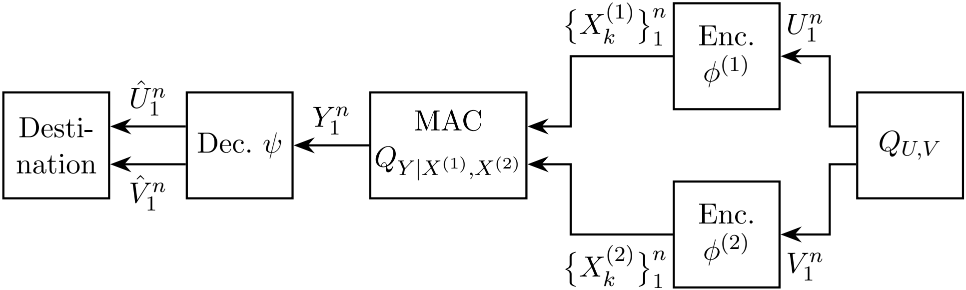
$R^{(2)}$
$\mathsf{C}\left(\frac{E^{(2)}}{\sigma^2}\right)$
C
$R^{(1)} + R^{(2)} = \mathsf{C}\left(\frac{E^{(1)}+E^{(2)}}{\sigma^2}\right)$
$\mathsf{C}\left(\frac{E^{(2)}}{E^{(1)}+\sigma^2}\right)$
B
$\mathsf{C}\left(\frac{E^{(1)}}{E^{(2)}+\sigma^2}\right)$
$\mathsf{C}\left(\frac{E^{(1)}}{\sigma^2}\right)$
$R^{(1)}$



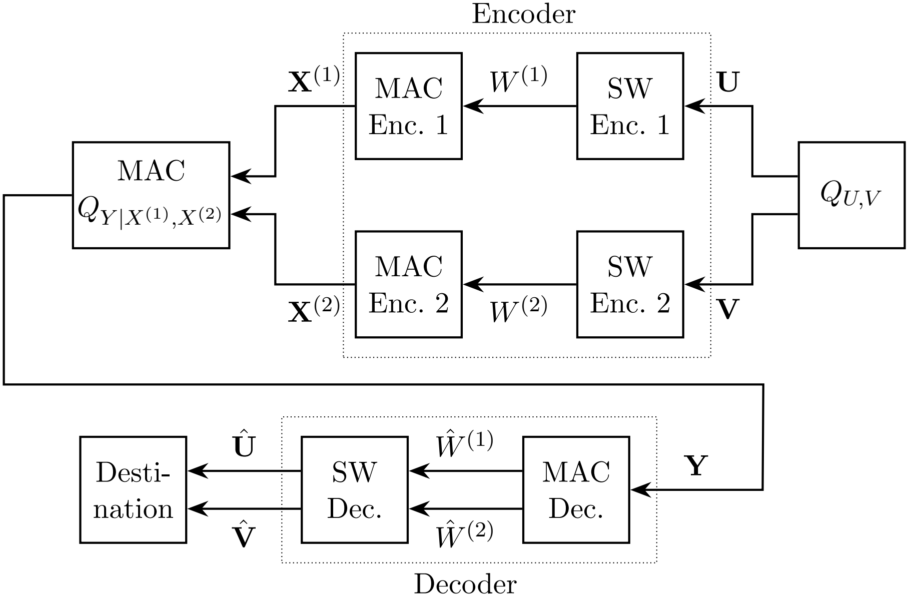
$R^{(2)}$
E
E
$R^{(1)}$

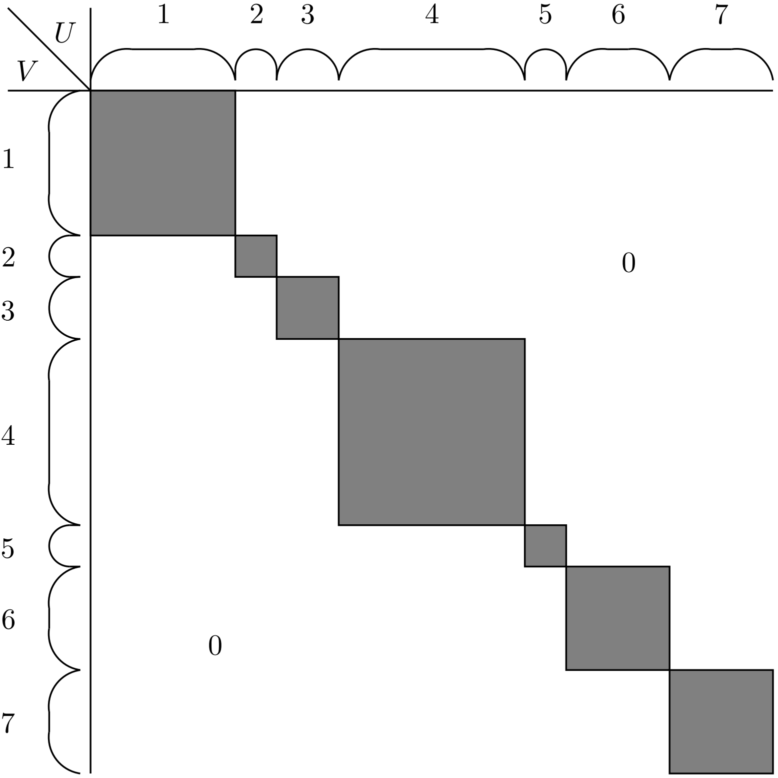


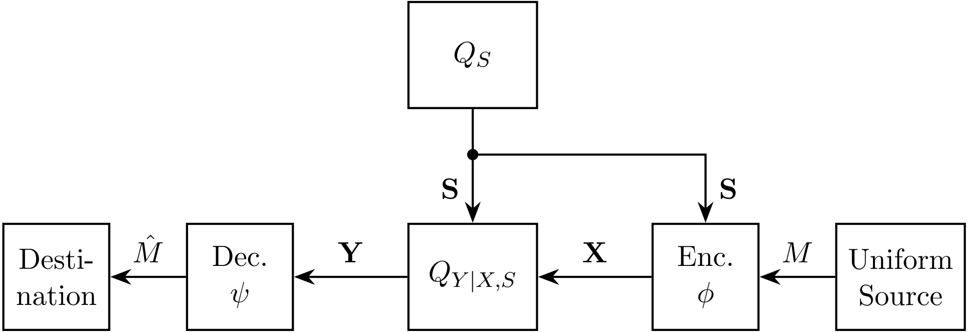
$R^{(2)}$
$\alpha = 0$
TDMA for α
from 0 to 1
$\alpha = \frac{E^{(1)}}{E^{(1)}+E^{(2)}}$
$\alpha = 1$
$R^{(1)}$



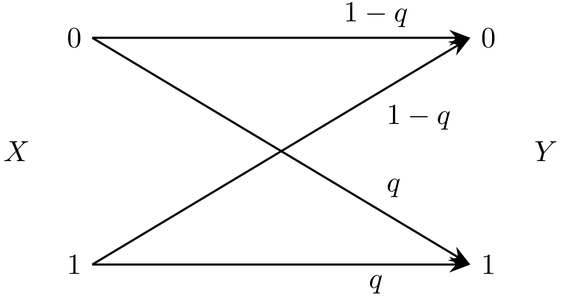
Destination
$\hat{U}_1^n$
$\hat{V}_1^n$
Dec. ψ
Y_1^n
MAC
$Q_{Y|X^{(1)},X^{(2)}}$
$\{X_k^{(1)}\}_1^n$
$\{X_k^{(2)}\}_1^n$
Enc. $\phi^{(1)}$
Enc. $\phi^{(2)}$
U_1^n
V_1^n
$Q_{U,V}$


Encoder
MAC
Enc. 1
SW
Enc. 1
$\mathbf{X}^{(1)}$
$W^{(1)}$
$\mathbf{U}$
MAC
$Q_{Y|X^{(1)},X^{(2)}}$
$Q_{U,V}$
MAC
Enc. 2
SW
Enc. 2
$\mathbf{X}^{(2)}$
$W^{(2)}$
$\mathbf{V}$
Desti-
nation
$\hat{\mathbf{U}}$
$\hat{\mathbf{V}}$
SW
Dec.
$\hat{W}^{(1)}$
$\hat{W}^{(2)}$
MAC
Dec.
$\mathbf{Y}$
Decoder

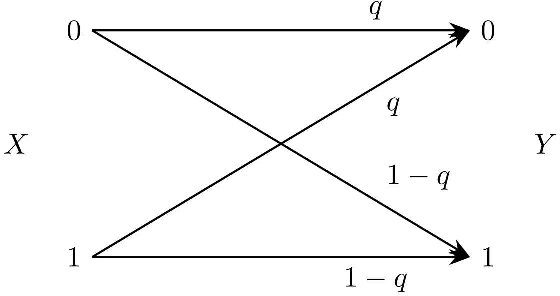

U
V
1
2
3
4
5
6
7
1
2
3
4
5
6
7
0
0



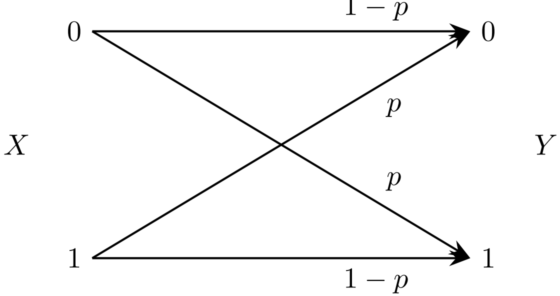
Q_S
$\mathbf{S}$
$\mathbf{S}$
Desti-
nation
$\hat{M}$
Dec.
ψ
$\mathbf{Y}$
$Q_{Y|X,S}$
$\mathbf{X}$
Enc.
ϕ
M
Uniform
Source



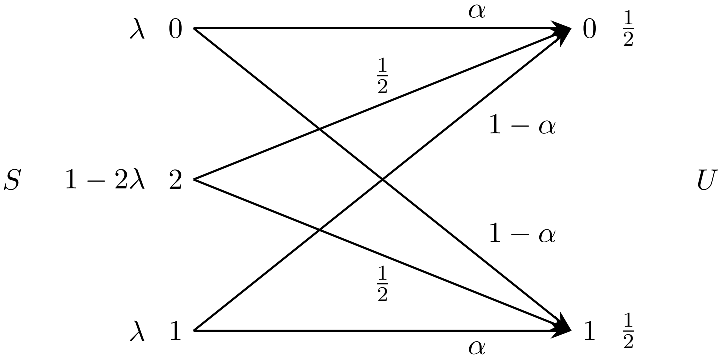
1 − q
0
0
1 − q
X
Y
q
1
1
q

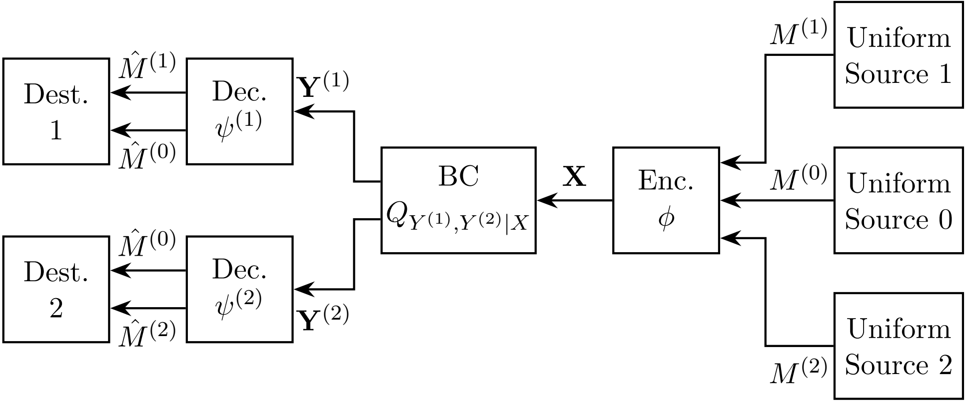


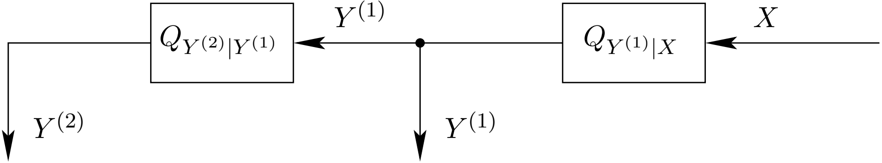
q
0
0
q
X
Y
$1-q$
1
1
$1-q$



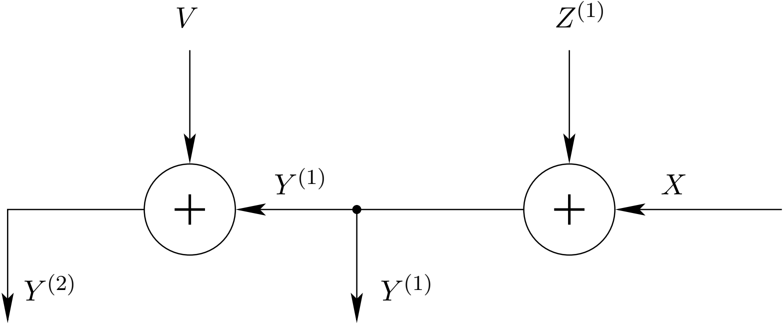
$1-p$
0
0
p
X
Y
p
1
1
$1-p$

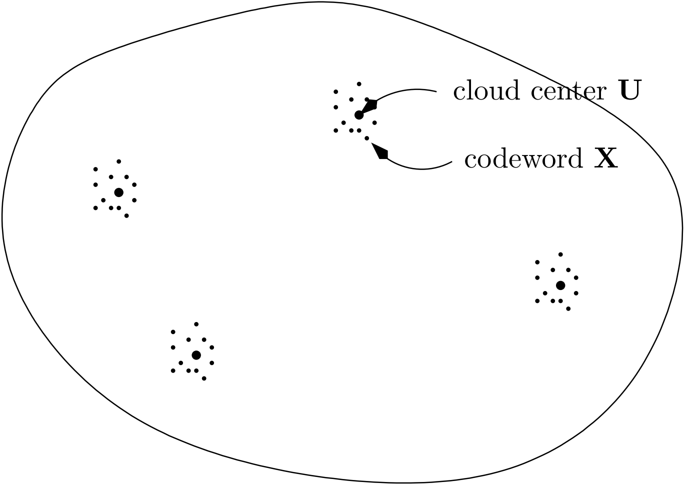

S
λ 0
$1-2\lambda$ 2
λ 1
α
$\frac{1}{2}$
$1-\alpha$
$1-\alpha$
$\frac{1}{2}$
α
0 $\frac{1}{2}$
1 $\frac{1}{2}$
U

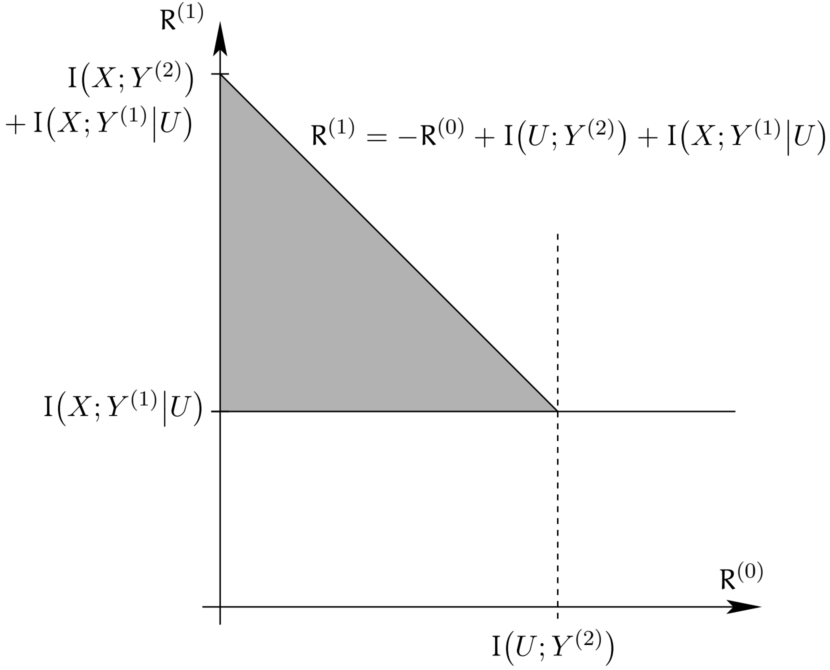

Dest.
1
Dec.
$\psi^{(1)}$
$\hat{M}^{(1)}$
$\hat{M}^{(0)}$
$\mathbf{Y}^{(1)}$
Dest.
2
Dec.
$\psi^{(2)}$
$\hat{M}^{(0)}$
$\hat{M}^{(2)}$
$\mathbf{Y}^{(2)}$
BC
$Q_{Y^{(1)},Y^{(2)}|X}$
$\mathbf{X}$
Enc.
ϕ
$M^{(1)}$
Uniform
Source 1
$M^{(0)}$
Uniform
Source 0
$M^{(2)}$
Uniform
Source 2



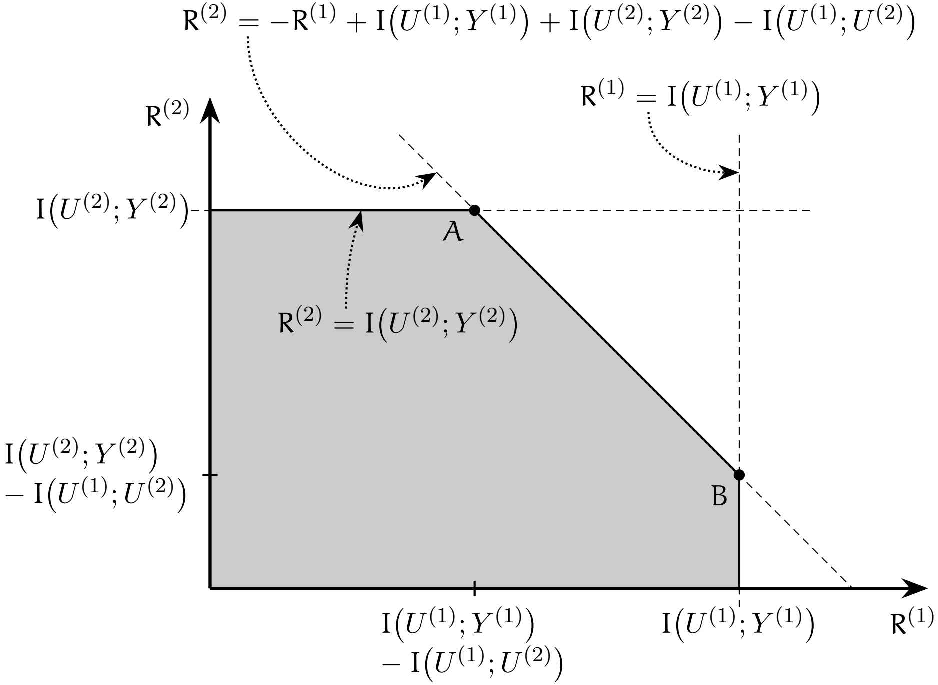
$Q_{Y^{(2)}|Y^{(1)}}$
$Y^{(1)}$
$Q_{Y^{(1)}|X}$
X
$Y^{(2)}$
$Y^{(1)}$


V
$Z^{(1)}$
$+$
$Y^{(1)}$
$+$
X
$Y^{(2)}$
$Y^{(1)}$

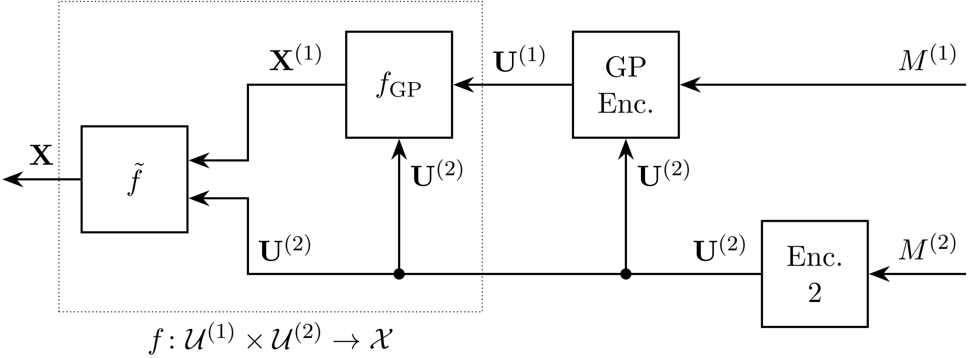

cloud center **U**
codeword **X**

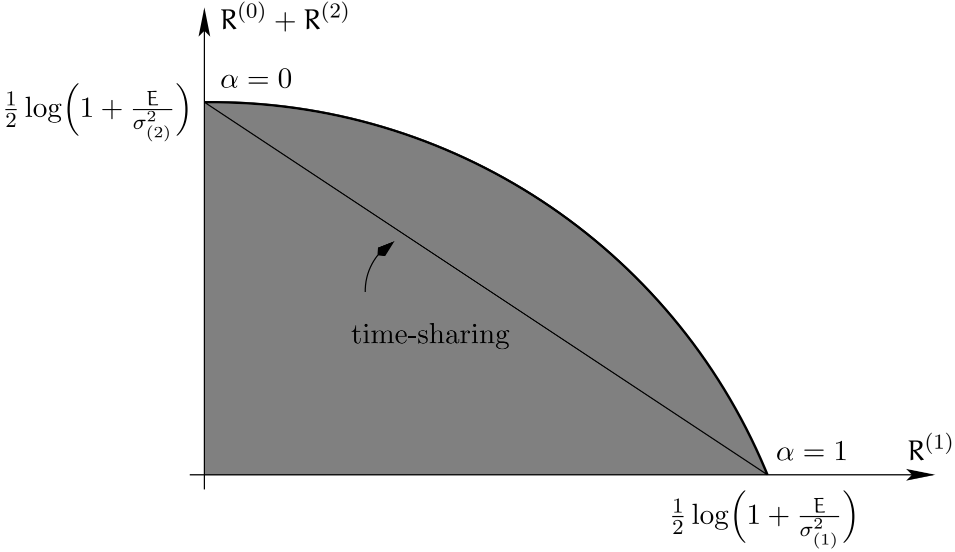


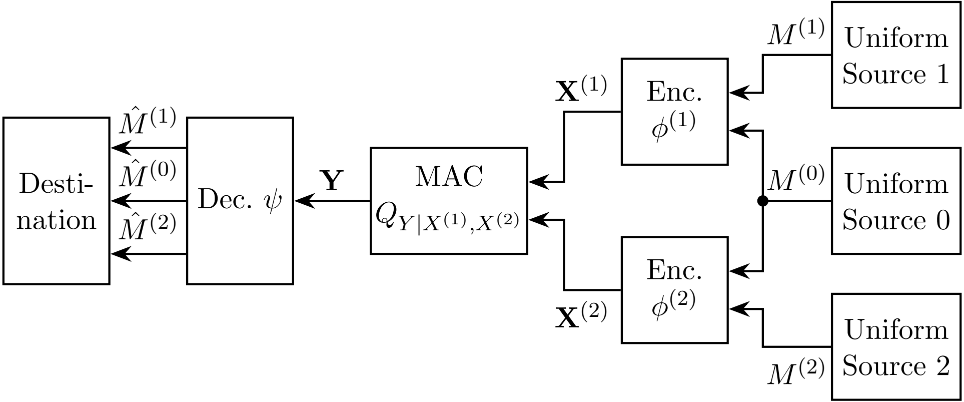
$\mathsf{R}^{(1)}$
$\mathrm{I}\bigl(X;Y^{(2)}\bigr)$
$+\,\mathrm{I}\bigl(X;Y^{(1)}\big|U\bigr)$
$\mathsf{R}^{(1)} = -\mathsf{R}^{(0)} + \mathrm{I}\bigl(U;Y^{(2)}\bigr) + \mathrm{I}\bigl(X;Y^{(1)}\big|U\bigr)$
$\mathrm{I}\bigl(X;Y^{(1)}\big|U\bigr)$
$\mathsf{R}^{(0)}$
$\mathrm{I}\bigl(U;Y^{(2)}\bigr)$



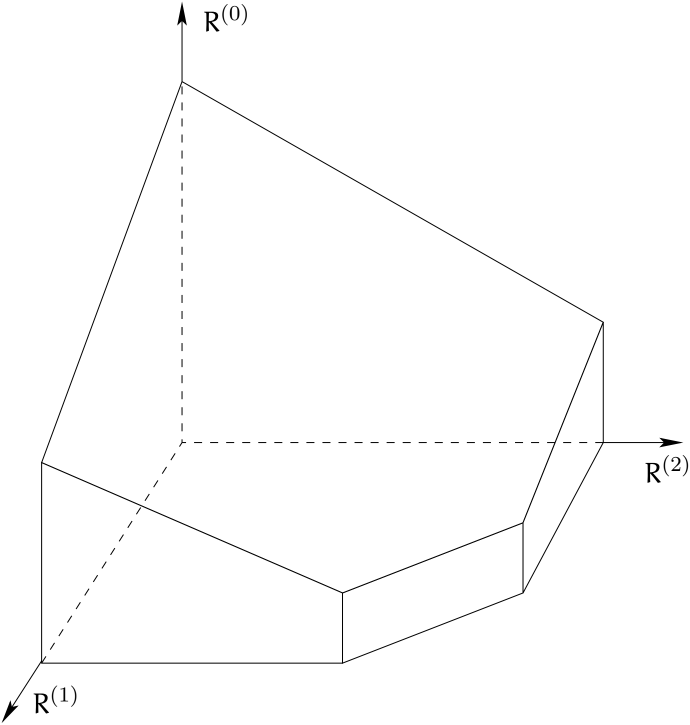
R(2) = −R(1) + I(U(1); Y(1)) + I(U(2); Y(2)) − I(U(1); U(2))
R(1) = I(U(1); Y(1))
R(2)
I(U(2); Y(2))
A
R(2) = I(U(2); Y(2))
I(U(2); Y(2))
− I(U(1); U(2))
B
I(U(1); Y(1))
− I(U(1); U(2))
I(U(1); Y(1))
R(1)

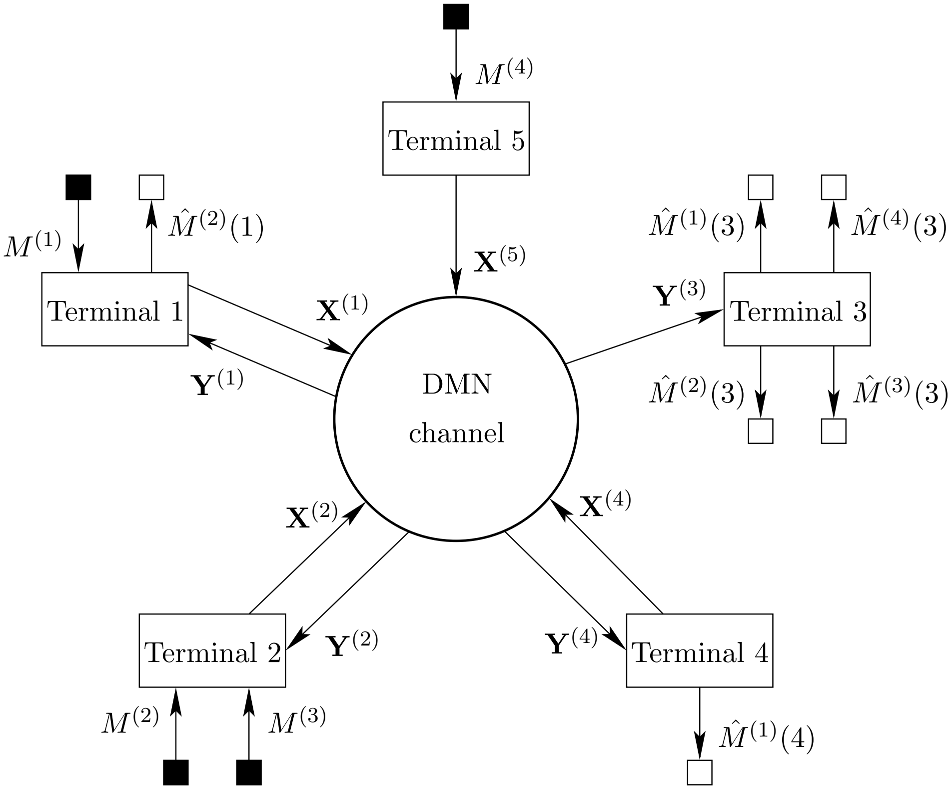

$\mathbf{X}$
$\tilde{f}$
$\mathbf{X}^{(1)}$
f_{GP}
$\mathbf{U}^{(1)}$
GP Enc.
$M^{(1)}$
$\mathbf{U}^{(2)}$
$\mathbf{U}^{(2)}$
$\mathbf{U}^{(2)}$
$\mathbf{U}^{(2)}$
Enc. 2
$M^{(2)}$
$f : \mathcal{U}^{(1)} \times \mathcal{U}^{(2)} \to \mathcal{X}$

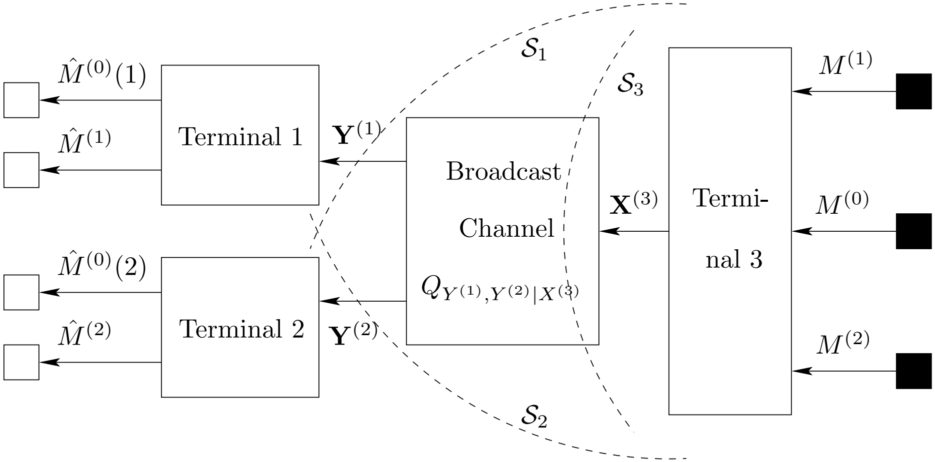

$\mathsf{R}^{(0)} + \mathsf{R}^{(2)}$
$\alpha = 0$
$\frac{1}{2}\log\left(1 + \frac{\mathsf{E}}{\sigma_{(2)}^2}\right)$
time-sharing
$\alpha = 1$
$\mathsf{R}^{(1)}$
$\frac{1}{2}\log\left(1 + \frac{\mathsf{E}}{\sigma_{(1)}^2}\right)$



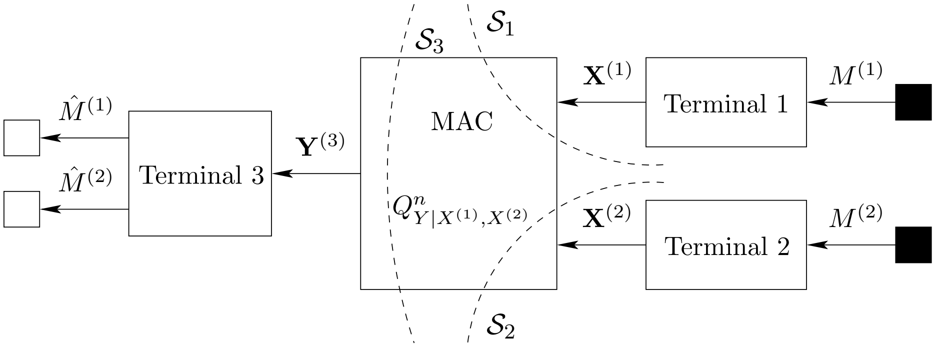
Destination
$\hat{M}^{(1)}$
$\hat{M}^{(0)}$
$\hat{M}^{(2)}$
Dec. ψ
$\mathbf{Y}$
MAC
$Q_{Y|X^{(1)},X^{(2)}}$
$\mathbf{X}^{(1)}$
$\mathbf{X}^{(2)}$
Enc. $\phi^{(1)}$
Enc. $\phi^{(2)}$
$M^{(1)}$
$M^{(0)}$
$M^{(2)}$
Uniform Source 1
Uniform Source 0
Uniform Source 2


$R^{(0)}$
$R^{(2)}$
$R^{(1)}$

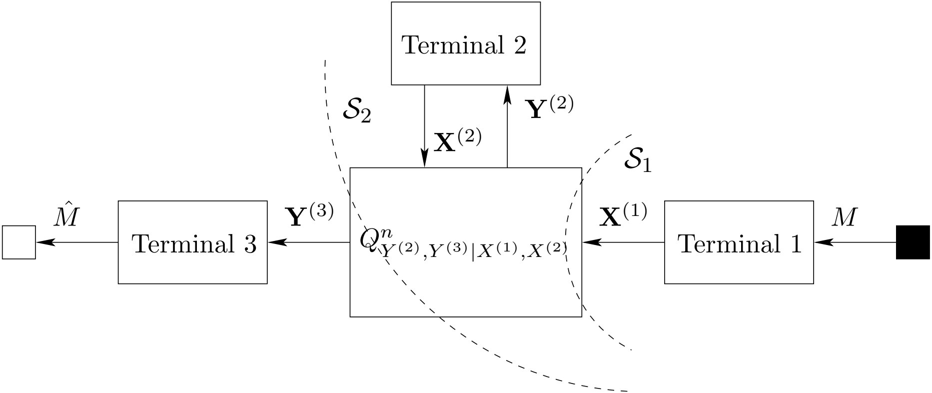

$M^{(4)}$
Terminal 5
$\mathbf{X}^{(5)}$
$M^{(1)}$
$\hat{M}^{(2)}(1)$
Terminal 1
$\mathbf{X}^{(1)}$
$\mathbf{Y}^{(1)}$
$\hat{M}^{(1)}(3)$
$\hat{M}^{(4)}(3)$
$\mathbf{Y}^{(3)}$
Terminal 3
$\hat{M}^{(2)}(3)$
$\hat{M}^{(3)}(3)$
DMN
channel
$\mathbf{X}^{(2)}$
$\mathbf{X}^{(4)}$
Terminal 2
$\mathbf{Y}^{(2)}$
$\mathbf{Y}^{(4)}$
Terminal 4
$M^{(2)}$
$M^{(3)}$
$\hat{M}^{(1)}(4)$



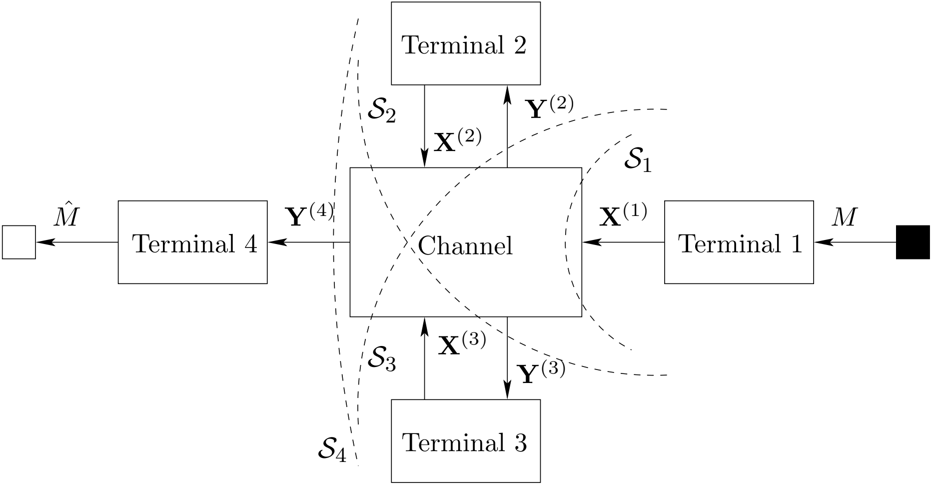
$\mathcal{S}_1$
$\mathcal{S}_3$
$\mathcal{S}_2$
$\hat{M}^{(0)}(1)$
$\hat{M}^{(1)}$
$\hat{M}^{(0)}(2)$
$\hat{M}^{(2)}$
Terminal 1
Terminal 2
$\mathbf{Y}^{(1)}$
$\mathbf{Y}^{(2)}$
Broadcast
Channel
$Q_{Y^{(1)},Y^{(2)}|X^{(3)}}$
$\mathbf{X}^{(3)}$
Termi-
nal 3
$M^{(1)}$
$M^{(0)}$
$M^{(2)}$

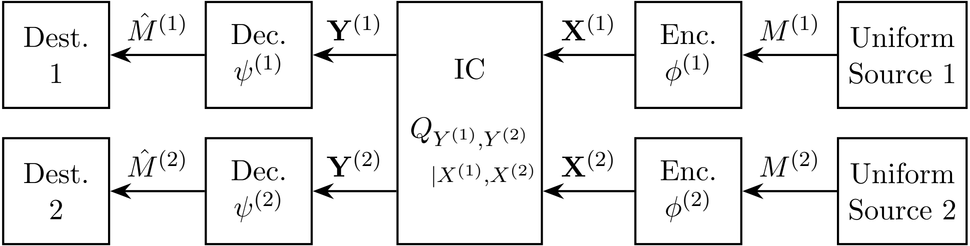


$\mathcal{S}_1$
$\mathcal{S}_3$
$\mathbf{X}^{(1)}$
$M^{(1)}$
Terminal 1
$\hat{M}^{(1)}$
MAC
$\mathbf{Y}^{(3)}$
Terminal 3
$\hat{M}^{(2)}$
$Q^n_{Y|X^{(1)},X^{(2)}}$
$\mathbf{X}^{(2)}$
$M^{(2)}$
Terminal 2
$\mathcal{S}_2$

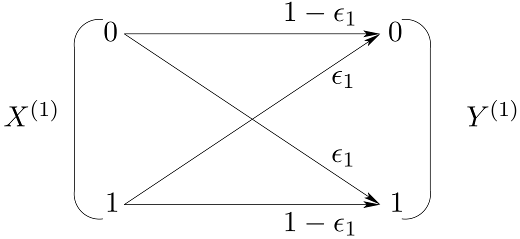


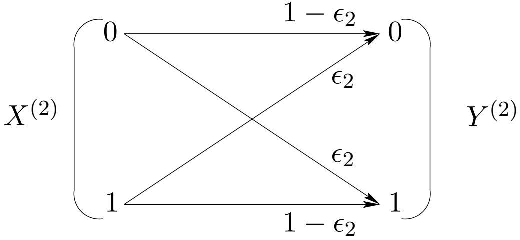
Terminal 2
$\mathcal{S}_2$
$\mathbf{Y}^{(2)}$
$\mathbf{X}^{(2)}$
$\mathcal{S}_1$
$\hat{M}$
$\mathbf{Y}^{(3)}$
$\mathbf{X}^{(1)}$
M
Terminal 3
$Q^n_{Y^{(2)},Y^{(3)}|X^{(1)},X^{(2)}}$
Terminal 1

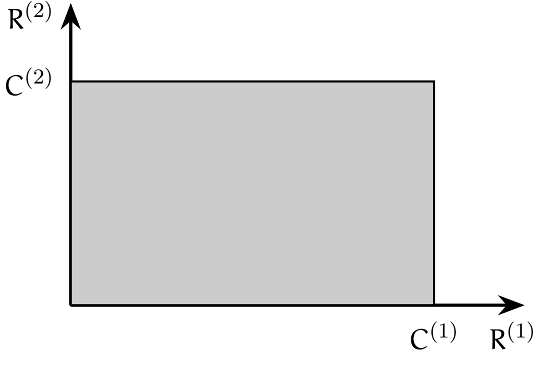

Terminal 2
$\mathbf{Y}^{(2)}$
$\mathcal{S}_2$
$\mathbf{X}^{(2)}$
$\mathcal{S}_1$
$\hat{M}$
$\mathbf{Y}^{(4)}$
$\mathbf{X}^{(1)}$
M
Terminal 4
Channel
Terminal 1
$\mathbf{X}^{(3)}$
$\mathcal{S}_3$
$\mathbf{Y}^{(3)}$
$\mathcal{S}_4$
Terminal 3

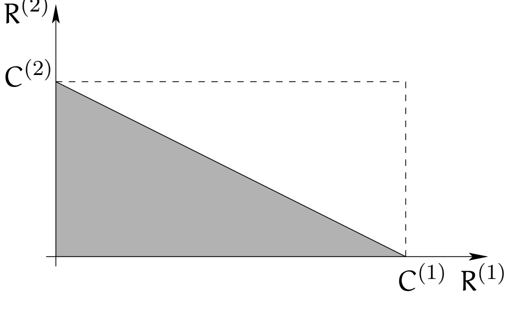


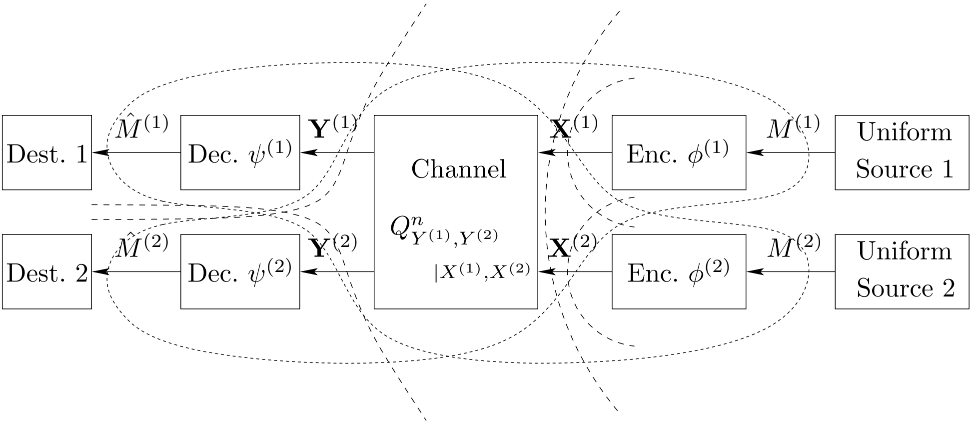
Dest.
1
$\hat{M}^{(1)}$
Dec.
$\psi^{(1)}$
$\mathbf{Y}^{(1)}$
IC
$Q_{Y^{(1)},Y^{(2)}|X^{(1)},X^{(2)}}$
$\mathbf{X}^{(1)}$
Enc.
$\phi^{(1)}$
$M^{(1)}$
Uniform
Source 1
Dest.
2
$\hat{M}^{(2)}$
Dec.
$\psi^{(2)}$
$\mathbf{Y}^{(2)}$
$\mathbf{X}^{(2)}$
Enc.
$\phi^{(2)}$
$M^{(2)}$
Uniform
Source 2


$1-\epsilon_1$
0
0
ϵ_1
$X^{(1)}$
$Y^{(1)}$
ϵ_1
1
1
$1-\epsilon_1$

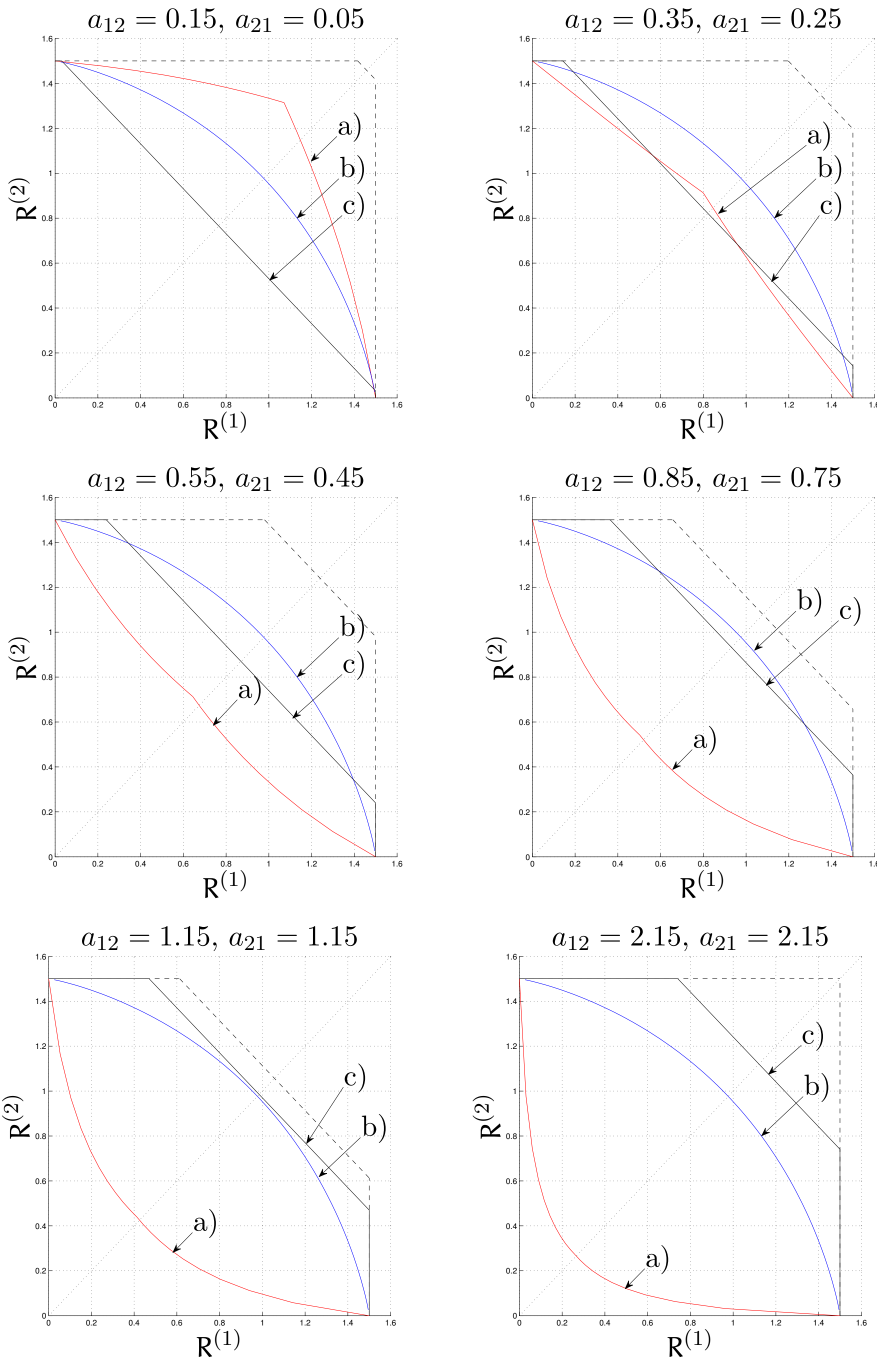

$1-\epsilon_2$
0
0
ϵ_2
$X^{(2)}$
$Y^{(2)}$
ϵ_2
1
1
$1-\epsilon_2$

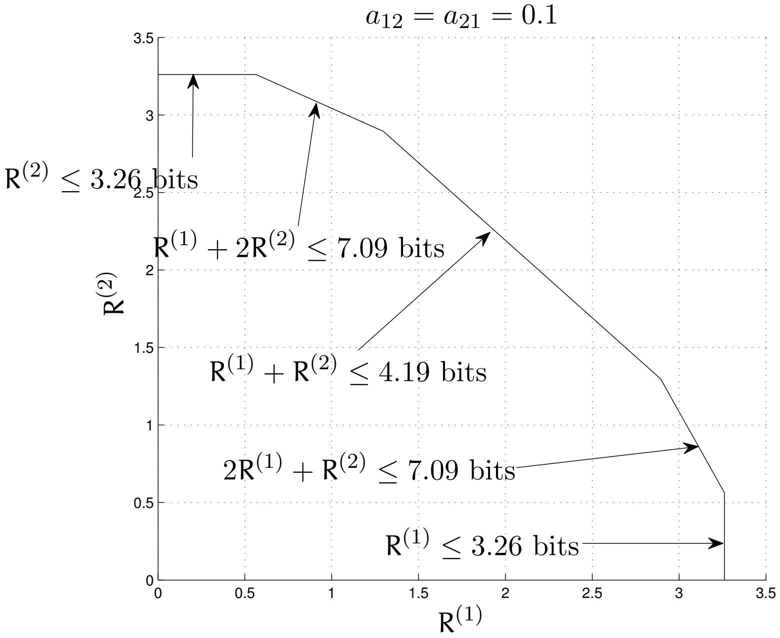

$R^{(2)}$
$C^{(2)}$
$C^{(1)}$
$R^{(1)}$

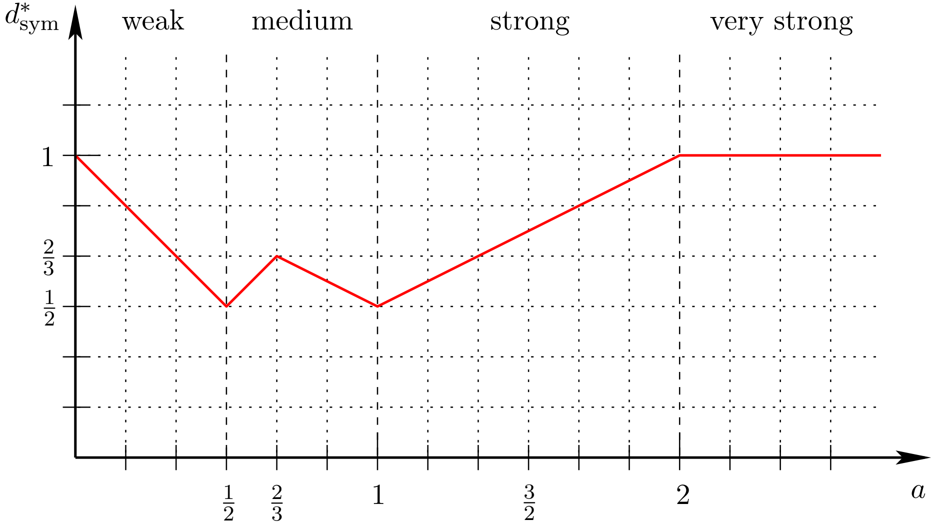

$R^{(2)}$
$C^{(2)}$
$C^{(1)}$
$R^{(1)}$



Dest. 1
$\hat{M}^{(1)}$
Dec. $\psi^{(1)}$
$\mathbf{Y}^{(1)}$
Channel
$Q^n_{Y^{(1)},Y^{(2)}|X^{(1)},X^{(2)}}$
$\mathbf{X}^{(1)}$
Enc. $\phi^{(1)}$
$M^{(1)}$
Uniform Source 1
Dest. 2
$\hat{M}^{(2)}$
Dec. $\psi^{(2)}$
$\mathbf{Y}^{(2)}$
$\mathbf{X}^{(2)}$
Enc. $\phi^{(2)}$
$M^{(2)}$
Uniform Source 2


$a_{12} = 0.15,\ a_{21} = 0.05$
$a_{12} = 0.35,\ a_{21} = 0.25$
$a_{12} = 0.55,\ a_{21} = 0.45$
$a_{12} = 0.85,\ a_{21} = 0.75$
$a_{12} = 1.15,\ a_{21} = 1.15$
$a_{12} = 2.15,\ a_{21} = 2.15$
$R^{(1)}$
$R^{(2)}$
a)
b)
c)


$a_{12} = a_{21} = 0.1$
3.5
3
2.5
2
1.5
1
0.5
0
0
0.5
1
1.5
2
2.5
3
3.5
$\mathsf{R}^{(2)}$
$\mathsf{R}^{(1)}$
$\mathsf{R}^{(2)} \leq 3.26$ bits
$\mathsf{R}^{(1)} + 2\mathsf{R}^{(2)} \leq 7.09$ bits
$\mathsf{R}^{(1)} + \mathsf{R}^{(2)} \leq 4.19$ bits
$2\mathsf{R}^{(1)} + \mathsf{R}^{(2)} \leq 7.09$ bits
$\mathsf{R}^{(1)} \leq 3.26$ bits


d^*_{sym}
weak
medium
strong
very strong
1
$\frac{2}{3}$
$\frac{1}{2}$
$\frac{1}{2}$
$\frac{2}{3}$
1
$\frac{3}{2}$
2
a