Convex-Programming Bounds on the Capacity of Flat-Fading Channels

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Abstract — **We propose a technique to derive analytic upper bounds on channel capacity via its dual expression. This is demonstrated on single- and multi-antenna flat fading channels where the receiver has no side information regarding the pathwise realization of the fading process. The results indicate that the capacity of such channels typically grows double-logarithmically in the SNR and not logarithmically as the piece-wise constant fading models predict.**

I. The Dual Expression

For any input distribution Q to a DMC $W(y|x)$ and any distribution $R(y)$ on its output the mutual information satisfies [1]:

$$I(Q,W) \leq \sum_{x\in\mathcal{X}} Q(x) D\big(W(\cdot|x)\|R(\cdot)\big). \tag{1}$$

This inequality can be shown to hold for more general alphabets, and can also be used (either directly or via Lagrange multipliers) to address input constraints. As we shall demonstrate using some examples of flat fading channels, a judicious choice of the probability measure $R(\cdot)$ may lead to useful analytic upper bounds on channel capacity.

II. Channel Model

We consider a channel with n_T transmit antennae and n_R receive antennae whose time-k output $\mathbf{Y}_k \in \mathbb{C}^{n_\mathrm{R}}$ is given by

$$\mathbf{Y}_k = \mathbb{H}_k \mathbf{x}_k + \mathbf{Z}_k = \sum_{t=1}^{n_\mathrm{T}} x_k^{(t)} \mathbf{H}_k^{(t)} + \mathbf{Z}_k, \tag{2}$$

where $\mathbf{x}_k = (x_k^{(1)}, \ldots, x_k^{(n_\mathrm{T})})^\mathsf{T} \in \mathbb{C}^{n_\mathrm{T}}$ denotes the time-k input vector, the vectors $\{\mathbf{Z}_k\}$ are spatially and temporally white zero-mean variance-$\mathcal{N}$ circularly-symmetric Gaussians, and the components of the random matrices $\{\mathbb{H}_k\}$ are jointly stationary and ergodic stochastic processes independent of $\{\mathbf{Z}_k\}$. We denote the capacity of this channel under the average power constraint $\mathsf{E}\left[\mathbf{X}_k^\dagger \mathbf{X}_k\right] \leq \mathcal{E}_\mathrm{s}$ by $C(\mathcal{E}_\mathrm{s})$.

Theorem 1. (Villa Serbelloni): *If $\{\mathbb{H}_k\}$ are IID, $\mathsf{E}\left[\operatorname{tr}(\mathbb{H}_k^\dagger \mathbb{H}_k)\right] < \infty$, and the differential entropy $h(\mathbb{H}_k)$ exists and is greater than $-\infty$, then*

$$\limsup_{\mathcal{E}_s\uparrow\infty} \left\{ C(\mathcal{E}_s) - \log\left(1 + \log\left(1 + \frac{\mathcal{E}_s}{\mathcal{N}}\right)\right)\right\} < \infty. \tag{3}$$

This theorem extends to Gaussian fading with memory:

Theorem 2. *Suppose now that the components of $\{\mathbb{H}_k - \mathsf{E}[\mathbb{H}_k]\}$ form jointly circularly-symmetric Gaussian stochastic processes that are regular in the sense that the error-covariance in estimating $\mathbb{H}_k$ from its infinite past is non-singular. Then (3) holds.*

The proofs are based on the following convex-programming bound for IID fading. (All logarithms are natural.)

$$\begin{aligned} C(\mathcal{E}_\mathrm{s}) \leq \inf_{\alpha>0} \sup \Big\{ & \log\Gamma(\alpha) - \alpha\log\alpha \\ & + m\log\pi - \log\Gamma(m) + m\mathsf{E}\left[\log\|\mathbf{Y}\|^2\right] - h(\mathbf{Y}|\mathbf{X}) \\ & + \alpha\Big(1 + \log\mathsf{E}\left[\|\mathbf{Y}\|^2\right] - \mathsf{E}\left[\log\|\mathbf{Y}\|^2\right]\Big)\Big\}, \end{aligned} \tag{4}$$

where the supremum is over all input distributions satisfying the average power constraint $\mathsf{E}\left[\|\mathbf{X}\|^2\right] \leq \mathcal{E}_\mathrm{s}$.

With the aid of (4) one can also obtain bounds on the capacity of an IID multi-antenna Ricean channel where the components of $\mathbb{H} - \mathsf{E}[\mathbb{H}]$ are IID $\mathcal{N}_\mathbb{C}(0,1)$, namely

$$\begin{aligned} C(\mathcal{E}_\mathrm{s}) \leq \inf_{0<\alpha\leq m} \Bigg\{ & \log\Gamma(\alpha) - \alpha\log\alpha - m - \log\Gamma(m) \\ & + \alpha\left[1 + \log\left(m\Big(1 + \frac{\mathcal{E}_\mathrm{s}}{\mathcal{N}}\Big) + \frac{\delta\mathcal{E}_\mathrm{s}}{\mathcal{N}}\right)\right] \\ & + (m-\alpha)\Bigg[\log\left(\frac{\delta\mathcal{E}_\mathrm{s}}{\mathcal{E}_\mathrm{s}+\mathcal{N}}\right) - \operatorname{Ei}\left(-\frac{\delta\mathcal{E}_\mathrm{s}}{\mathcal{E}_\mathrm{s}+\mathcal{N}}\right) \\ & + \sum_{j=1}^{m-1}\left(-\frac{\delta\mathcal{E}_\mathrm{s}}{\mathcal{E}_\mathrm{s}+\mathcal{N}}\right)^{-j}\Bigg(e^{-\frac{\delta\mathcal{E}_\mathrm{s}}{\mathcal{E}_\mathrm{s}+\mathcal{N}}}(j-1)! \\ & - \frac{(m-1)!}{j(m-1-j)!}\Bigg)\Bigg]\Bigg\}, \end{aligned} \tag{5}$$

where δ is the maximum eigenvalue of $\mathsf{E}[\mathbb{H}]^\dagger\,\mathsf{E}[\mathbb{H}]$.

In the single-antenna IID Rayleigh fading case this bound can be further tightened to

$$\begin{aligned} C(\mathcal{E}_\mathrm{s}) \leq \inf_{\alpha,\beta>0}\inf_{\delta\geq 0} \Bigg\{ & -1 - \alpha\log\frac{\delta}{\beta} + \log\Gamma\left(\alpha, \frac{\delta}{\beta}\right) + \frac{\mathcal{E}_\mathrm{s}+\mathcal{N}}{\beta} \\ & + \frac{\delta}{\beta} + \log\frac{\delta}{\mathcal{N}} + (\alpha-1)e^{\frac{\delta}{\mathcal{N}}}\operatorname{Ei}\left(-\frac{\delta}{\mathcal{N}}\right)\Bigg\}, \end{aligned} \tag{6}$$

which gives the precise asymptotics

$$C(\mathcal{E}_\mathrm{s}) = \log\big(1 + \log(1 + \mathcal{E}_\mathrm{s}/\mathcal{N})\big) - \gamma - 1 + o(1), \tag{7}$$

where γ is Euler's constant, and the term $o(1)$ tends to zero as $\mathcal{E}_\mathrm{s}/\mathcal{N}$ tends to infinity.

References

[1] R. G. Gallager, *Information theory and reliable communication.* Exercise 4.17, pp. 524–525. New York: Wiley, 1968.

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