The Asymptotic Capacity of the Discrete-Time Poisson Channel

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Abstract — **The large-inputs asymptotic capacity of a peak and average power limited discrete-time Poisson channel is derived using a new firm (non-asymptotic) lower bound and an asymptotic upper bound. The latter upper bound is based on the dual expression for channel capacity and the notion of capacity-achieving input distributions that escape to infinity.**

I. Introduction

We consider a memoryless discrete-time channel whose output Y takes value in the set of non-negative integers $\mathbb{Z}^+$ and whose input takes value in the set of non-negative reals $\mathbb{R}^+$. Conditional on the input $x \geq 0$, the output has a Poisson distribution of mean $x + \lambda_0$, where λ_0 is some non-negative constant. Thus

$$\Pr[Y{=}y \mid X{=}x] = e^{-(x+\lambda_0)} \frac{(x+\lambda_0)^y}{y!}, \quad y \in \mathbb{Z}^+, x \geq 0. \quad (1)$$

This channel is often used to model pulse-amplitude modulated optical communication with a direct-detection receiver [1]. Here the input x is proportional to the product of the transmitted light intensity by the pulse duration; λ_0 similarly models the time-intensity product of the background radiation ("dark current"); and the output Y models the number of photons arriving at the receiver during the pulse duration. An average power constraint on the transmitter is accounted for by the average input constraint $\mathsf{E}[X] \leq \mathcal{P}$ and a peak power constraint by $0 \leq X \leq \mathsf{A}$. We use $0 < \alpha \leq 1$ to denote the average-to-peak ratio $\alpha = \mathcal{P}/\mathsf{A}$. The case $\alpha = 1$ corresponds to the absence of an average power constraint, whereas $\alpha \ll 1$ corresponds to a very weak peak power constraint.

No analytic expression for the capacity of the Poisson channel is known. In [1] it was shown that capacity-achieving input distributions are discrete with a finite number of mass points that increases to infinity as the constraints are relaxed. Some bounds on capacity were reported in [2]. Here we present an improved lower bound to channel capacity and an asymptotic — peak and average powers tending to infinity with their ratio held fixed — upper bound. The upper and lower bounds asymptotically coincide, thus yielding the asymptotic expansion for channel capacity.

II. The Asymptotic Expansion

We begin with the case where only the average power constraint $\mathsf{E}[X] \leq \mathcal{P}$ is imposed. The capacity $C(\mathcal{P})$ is then

$$C(\mathcal{P}) = \frac{1}{2}\log \mathcal{P} + o(1), \quad (2)$$

where the error term $o(1)$ tends to zero as $\mathcal{P} \to \infty$.

Next consider the case where both average and peak power constraints are imposed. Holding the average-to-peak ratio α fixed, we distinguish between two cases: For $\alpha \geq 1/3$ (including the peak constraint only case $\alpha = 1$) the capacity $C(\mathsf{A}, \mathcal{P})$ is given by

$$C(\mathsf{A}, \mathcal{P}) = \frac{1}{2}\log \mathsf{A} - \frac{1}{2}\log\frac{\pi e}{2} + o(1), \quad \frac{1}{3} \leq \alpha \leq 1, \quad (3)$$

where the $o(1)$ term tends to zero as $\mathcal{P}$ and A tend to infinity with their ratio $\mathcal{P}/\mathsf{A}$ held fixed at a value between 1/3 and 1.

When $0 < \alpha < 1/3$ we have

$$C(\mathsf{A}, \mathcal{P}) = \frac{1}{2}\log \mathsf{A} + \alpha u - u - \log\left(\frac{1}{2} - \alpha u\right) - \frac{1}{2}\log 2\pi e + o(1), \quad (4)$$

where u is the non-zero solution to

$$\sqrt{\pi}\,\mathrm{erf}\left(\sqrt{u}\right)\left(\frac{1}{2} - \alpha u\right) - \sqrt{u}e^{-u} = 0, \quad (5)$$

and the $o(1)$ error term tends to zero as the average and peak powers tend to infinity with their ratio held fixed at α.

III. The Main Tools

Capacity is lower bounded by lower bounding $H(Y)$ and upper bounding $H(Y|X)$ in terms of the input X. The latter is based on the conditional variance of Y [3, Theorem 16.3.3]:

$$H(Y|X = x) \leq \frac{1}{2}\log 2\pi e\left(x + \lambda_0 + \frac{1}{12}\right). \quad (6)$$

As to the former, we lower bound $H(Y)$ using the following:

Lemma 1. *Let X be a non-negative random variable of finite expectation, and let Y be related to X according to the law (1). Then the entropy $H(Y)$ of Y is lower bounded by the differential entropy $h(X)$ of X.*

To derive the asymptotic upper bound on channel capacity we first note that no loss of information occurs if we add to the output Y an independent random variable U that is uniformly distributed over $[0, 1)$ to form the modified output $\tilde{Y}$, which takes value in $\mathbb{R}^+$. The upper bound now follows using the dual expression of channel capacity [4] and the concept of *capacity-achieving input distributions that escape to infinity* [4], applied to the channel $x \mapsto \tilde{Y}$.

References

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