Optimal Ultra-Small Block-Codes on Two Special Binary Channels

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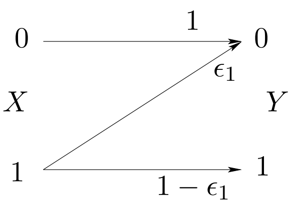
1 Introduction

Shannon proved in his ground-breaking work that it is possible to find an information transmission scheme that can transmit messages at arbitrarily small error probability as long as the transmission rate in **bits per channel use** is below the so-called **capacity** of the channel and the blocklength is very large. However, a large blocklength might not be practical. **The question now arises what we can theoretically say about the performance of communication systems with strongly restricted blocklength and their global optimal design.**

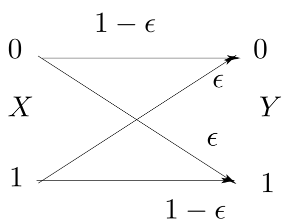
Here we focus on the number of messages up to at most four. Our goal is to try to find the optimal code structure (either linear or nonlinear) with respect to the **average error probability for any fixed blocklength** n on the binary symmetric channel (BSC) and the Z-channel. I.e., for a fixed n we try to design a code that minimizes the error probability among all possible codes of length n.

2 Channel Models

We consider two well-known binary channels: the **binary symmetric channel (BSC)** with crossover probability ϵ and the **Z-channel** with 1–0-crossover probability ϵ_1:



Z-channel



BSC

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3 Optimal Code Design

Definition 1. The **flip code of type** t for $t \in \{0, 1, \ldots, \lfloor \frac{n}{2} \rfloor\}$ is defined by the following codebook matrix $\mathscr{C}$ with $\mathrm{M} = 2$ codewords:

$$\mathscr{C}^{(2)} = \begin{pmatrix} 0 & \cdots & 0 & \overbrace{1 \;\; \cdots \;\; 1}^{t} \\ 1 & \cdots & 1 & 0 \;\; \cdots \;\; 0 \end{pmatrix}$$

A **weakly flip code of length** n for $\mathrm{M} = 3$ or $\mathrm{M} = 4$ codewords is defined by a codebook matrix that consists of n columns taken from the following candidate sets:

$$\left\{ \mathbf{c}_1^{(3)} \triangleq \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \mathbf{c}_2^{(3)} \triangleq \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{c}_3^{(3)} \triangleq \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

or

$$\left\{ \mathbf{c}_1^{(4)} \triangleq \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \mathbf{c}_2^{(4)} \triangleq \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{c}_3^{(4)} \triangleq \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\},$$

respectively.

Proposition 2. *For $\mathrm{M} = 2$ codewords and any blocklength n, on a BSC, the* **flip codes of type** *t for any $t \in \{0, 1, \ldots, \lfloor \frac{n}{2} \rfloor\}$ are all optimal, and on a Z-channel, the* **flip code of type** *0 is optimal.*

Theorem 3. *For $\mathrm{M} = 3$ or $\mathrm{M} = 4$ codewords and any blocklength n, on a Z-channel, an optimal choice for the codebook matrix is*

$$\mathscr{C}^{(\mathrm{M})} = \left(\mathbf{c}_1^{(\mathrm{M})} \, \mathbf{c}_2^{(\mathrm{M})} \, \mathbf{c}_1^{(\mathrm{M})} \, \mathbf{c}_2^{(\mathrm{M})} \, \mathbf{c}_1^{(\mathrm{M})} \, \mathbf{c}_2^{(\mathrm{M})} \, \cdots \right).$$

On a BSC, an optimal code for $\mathrm{M} = 3$ is the **weakly flip code** *constructed recursively by the following choice of candidate columns:*

$$\mathscr{C}^{(3)} = \left(\mathbf{c}_1^{(3)} \, \mathbf{c}_3^{(3)} \, \mathbf{c}_1^{(3)} \, \mathbf{c}_2^{(3)} \, \mathbf{c}_3^{(3)} \, \mathbf{c}_1^{(3)} \, \mathbf{c}_2^{(3)} \, \cdots \right);$$

an optimal linear code for $\mathrm{M} = 4$ is the **weakly flip code** *constructed recursively by the following choice of candidate columns:*

$$\mathscr{C}^{(4)} = \left(\mathbf{c}_1^{(4)} \, \mathbf{c}_3^{(4)} \, \mathbf{c}_1^{(4)} \, \mathbf{c}_2^{(4)} \, \mathbf{c}_3^{(4)} \, \mathbf{c}_1^{(4)} \, \mathbf{c}_2^{(4)} \, \cdots \right).$$