Sphere Covering for Poisson Processes

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Abstract—The geometric interpretation of sphere covering describing the rate distortion problem of a Gaussian source with the squared-error distortion measure is generalized to a Laplacian source and the $\ell_1$-distortion measure. Using additional constraints on the distortion measure, sphere covering is further generalized to exponential sources and to Poisson point processes.

I. INTRODUCTION

It is well-known that the rate-distortion problem of a Gaussian source and a squared-error distortion measure can be understood geometrically by counting the minimum number of small distortion balls required to completely cover the volume of the ball describing the possible outputs of the Gaussian source [1, Sec. 10.9], [2, Sec. 10.5]. This beautiful picture of sphere covering,\(^1\) however, is not restricted to the special case of Gaussian sources with an $\ell_2$-distortion measure, but can be generalized to sources and corresponding distortion measures in the general $\ell_p$-space. In this paper, we focus on the case for $p = 1$, i.e., a Laplacian source with the $\ell_1$-distortion measure, and then show how it can be adapted to find a corresponding geometric interpretation of the well-known rate-distortion function of a Poisson point process. The general case in $\ell_p$ is deferred to a later publication.

The remainder of this paper is structured as follows. After some definitions and comments about notation in Section II, we briefly recall the well-known sphere covering interpretation for the Gaussian source in Section III. In Section IV we then develop sphere covering in $\ell_1$ for a Laplacian source. Section V adapts the results from Section IV to allow the description of an exponential source (and a constrained distortion measure), and Section VI finally looks at the Poisson point processes. We focus on geometric proofs for the converse part of the rate-distortion problem and omit the achievability proofs. Note that all given lower bounds to the rate-distortion functions are tight.

II. NOTATION

The $\ell^*_p$-ball in $\mathbb{R}^n$ around the center $\hat{x} \in \mathbb{R}^n$ of radius $r > 0$ is given as

$$B^*_p(r, \hat{x}) \triangleq \left\{ x \in \mathbb{R}^n : \left( \sum_{i=1}^{n} |x_i - \hat{x}_i|^p \right)^{1/p} \leq r \right\}. \tag{1}$$

\(^1\)A similar geometric interpretation exists for the channel coding problem leading to sphere packing [3, Sec. 17.3], [2, Ch. 9].

and its surface, the $\ell^*_p$-sphere in $\mathbb{R}^n$, is

$$S^{n-1}_p(r, \hat{x}) \triangleq \left\{ x \in \mathbb{R}^n : \left( \sum_{i=1}^{n} |x_i - \hat{x}_i|^p \right)^{1/p} = r \right\}. \tag{2}$$

For the $n$-simplex in $\mathbb{R}^n$ around the center $\hat{x} \in \mathbb{R}^n$ we write

$$\triangle^{n-1}(r, \hat{x}) \triangleq \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} (x_i - \hat{x}_i) = r, \quad x_i - \hat{x}_i > 0, \forall i \in \{1, \ldots, n\} \right\}. \tag{3}$$

Sometimes we omit the second argument in the sets given in (1)–(3), in which case it is understood that a vector $0$ should be substituted, i.e., $B^*_p(r) \triangleq B^*_p(r, 0)$, $S^{n-1}_p(r) \triangleq S^{n-1}_p(r, 0)$, and $\triangle^{n-1}(r) \triangleq \triangle^{n-1}(r, 0)$.

We use $\text{vol}_n$ to denote the $n$-dimensional Lebesgue measure. Bold font denotes vectors (e.g., $x$ and $0$), and sets are in a calligraphic font $\mathcal{B}$. The logarithm $\log(\cdot)$ is to base 2.

III. RATE DISTORTION AND SPHERE COVERING IN $\ell_2$

We recall the familiar geometric picture for the Gaussian-quadratic rate-distortion problem based on sphere covering. For a Gaussian source of variance $\sigma^2$ and mean 0, the rate-distortion code with distortion $D$ yields a minimal rate given by taking the volume ratio of the source ball $B^*_2(\sqrt{n\sigma^2})$ and the distortion ball $B^*_2(\sqrt{nD})$ in $\mathbb{R}^n$, i.e.,

$$R_{\text{Gaussian}}(D) = \lim_{n \to \infty} \frac{1}{n} \log \frac{\text{vol}_n(B^*_2(\sqrt{n\sigma^2}))}{\text{vol}_n(B^*_2(\sqrt{nD}))}. \tag{4}$$

Note that the length-$n$ codewords that achieve (4) are the coordinates of the centers of the distortion balls. While we know that optimal codewords can be generated IID according to a Gaussian distribution of variance $\sigma^2 - D$ and mean 0, this fact cannot be deduced from the sphere-covering argument easily if we follow the geometric picture as given in Fig. 1(a), where the $n$-dimensional distortion balls cover the $n$-dimensional source ball.

To infer the generation of rate-distortion codewords, we need an alternative geometric picture, as already explored in [4, Sec. 1], which we depict in Fig. 1(b): Here we note that the source sequences generated by the Gaussian source actually lie within a thin layer close to the surface of the
source ball almost surely. Therefore, we aim to enclose\(^2\) the surface of the source ball entirely with a minimal number of distortion balls, i.e., cover the surface \(S_2^{n-1}(\sqrt{n\sigma^2})\) with the cross sections \(S_2^{n-1}(\sqrt{n\sigma^2}) \cap B_2^n(\sqrt{nD}, \hat{x}_m)\), where the codewords \(\hat{x}_m\) are selected such that each cross section is maximized. From Fig. 1(b), one immediately sees that all \(\hat{x}_m\) lie on the surface of an \(\ell_2^n\)-ball:

\[
\begin{align*}
\hat{x}_m &\in \arg\max_{\tilde{x}\in\mathbb{R}^n} \text{vol}_{n-1}(S_2^{n-1}(\sqrt{n\sigma^2}) \cap B_2^n(\sqrt{nD}, \tilde{x})) \\
&= S_2^{n-1}(\sqrt{n\sigma^2 - D}).
\end{align*}
\]

Figure 1. Geometric picture of \(\ell_2^n\) sphere covering: on the left all distortion balls cover the source ball, while on the right only the surface of the source ball is covered.

To this point, one may conjecture that the relationship between rate distortion and sphere covering exists beyond the well-known case for the Gaussian source in the \(\ell_2^n\)-space presented in this section. In the following sections we present our results for an analogous case of sphere covering with \(\ell_1^n\)-balls in \(\mathbb{R}^n\), define its corresponding rate-distortion problem, extend it to an exponential source, and finally link it to known rate-distortion functions for the Poisson process.

### IV. RATE DISTORTION AND SPHERE COVERING IN \(\ell_1^n\)

#### A. The Laplacian-\(\ell_1^n\) Rate-Distortion Problem

To generalize the ideas of Section III to the \(\ell_1^n\)-space we consider the Laplacian source

\[ X \sim P_X(x) \triangleq \frac{1}{2r_0} e^{-|x|/r_0} \tag{7} \]

and choose the \(\ell_1^n\)-distortion measure

\[ d(x, \hat{x}) \triangleq |x - \hat{x}|. \tag{8} \]

Then, by recalling the standard definition

\[ d(x, \hat{x}) \triangleq \frac{1}{n} \sum_{i=1}^{n} d(x_i, \hat{x}_i) = \frac{1}{n} \sum_{i=1}^{n} |x_i - \hat{x}_i|, \tag{9} \]

one immediately realizes that the distortion ball\(^3\) centered at \(\hat{x}\) is by definition an \(\ell_1^n\)-ball, i.e.,

\[ \{ x \in \mathbb{R}^n : d(x, \hat{x}) \leq D \} = B_1^n(nD, \hat{x}). \tag{10} \]

In a naive approach we can now argue that, for large \(n\), the source sequence \(X\) will lie with high probability within the \(\ell_1^n\)-ball of radius \(nr_0\), \(B_1^n(nr_0)\). Thus, by computing the ratio of the volume of the large radius-(\(nr_0\)) ball to the volume of the small radius-\((nD)\) balls, we can easily deduce the minimum number of codewords needed to cover the complete source ball:

\[ |C^n| \geq \frac{\text{vol}_n(B_1^n(nr_0))}{\text{vol}_n(B_1^n(nD))} = \left(\frac{r_0}{D}\right)^n \tag{11} \]

(see Fig. 2(a)). From this, the rate-distortion function (or rather a lower bound to it) can be derived using that the rate is defined as

\[ R = \frac{1}{n} \log |C^n|. \tag{12} \]

We would like to present a more accurate geometric picture by observing that the source sequences generated by \(P_X\) lie within a thin layer at the surface of the source ball \(B_1^n(nr_0)\) almost surely, since \(\frac{1}{n} \sum_{i=1}^{n} |x_i| \rightarrow r_0\) with probability 1 as \(n \rightarrow \infty\) [5, Sec. II].

Thus, reminiscent of Section III, we aim to enclose the surface of the source ball with a minimal number of distortion balls, i.e., we intersect the source \(\ell_1^n\)-sphere with many small distortion balls, creating tiles that cover the complete surface of the source ball, see Fig. 2(b).

**Theorem 1:** For a Laplacian source (7) and an \(\ell_1^n\)-distortion measure (8), define \(D \triangleq \lim_{n \rightarrow \infty} E[d(X, X)]\). Then the rate-distortion function is lower-bounded as

\[ R_{\text{Laplacian}}(D) \geq \begin{cases} \log \left( \frac{r_0}{D} \right) & \text{if } 0 < D \leq r_0, \\ 0 & \text{otherwise}. \end{cases} \tag{13} \]

**Proof:** This theorem follows from Theorem 3 below.

\(^2\)The phrase “enclose a surface” is used when we refer to covering an arbitrarily small inflation of this \((n - 1)\)-dimensional surface in \(\mathbb{R}^n\) for \(n\) sufficiently large.

\(^3\)Note that we follow common practice and talk about \(\ell_2^n\)-balls and \(\ell_1^n\)-spheres, even though the corresponding shape does not look like a traditional ball at all, but rather is a higher-dimensional version of the three-dimensional regular octahedron. See also Fig. 2.
B. Sphere Covering with $\ell^n_1$-Balls in $\mathbb{R}^n$

When investigating ways of how a small $\ell^n_1$-ball can cover a large $\ell^n_1$-sphere (see Fig. 2(b)), we will avoid distinguishing various different cases of covering facets, edges, or corners, but rather directly prove that the covered $(n-1)$-dimensional part of the large $\ell^n_1$-sphere is upper-bounded by the surface of the small $\ell^n_1$-ball.

Proposition 2: For radii $r_b > r_s$, and arbitrary $z$,
\[ \text{vol}_{n-1}(S^n_{1-1}(r_b) \cap B^n_1(r_s, z)) \leq \text{vol}_{n-1}(S^n_{1-1}(r_s, z)). \]

Proof: See Appendix A.

Theorem 3 (Asymptotic sphere covering in $\ell^n_1$): For $n \geq 2$, fix two radii $r_b > r_s > 0$ and denote by $C^n \triangleq \{\hat{x}_i \in \mathbb{R}^n\}$ a collection of centers of $\ell^n_1$-balls $B^n_1(r_s, \hat{x}_i)$. Then for any $C^n$ such that the union of radius-$r_b$ balls around the centers in $C^n$ do cover the complete surface of the radius-$r_b$ ball,
\[ S^n_{1-1}(r_b) \subseteq \bigcup_{x \in C^n} B^n_1(r_b, \hat{x}), \]
the following inequality holds:
\[ \lim_{n \to \infty} \frac{1}{n} \log |C^n| \geq \log \left( \frac{r_b}{r_s} \right). \]

Proof: The minimal size of $|C^n|$ is achieved when $\hat{x}_i$ are selected such that $\text{vol}_{n-1}(S^n_{1-1}(r_b) \cap B^n_1(r_s, \hat{x}))$ is maximized. From Proposition 2 we know that the latter is upper-bounded by the boundary measure of the small ball. Thus,
\[ |C^n| \geq \frac{\text{vol}_{n-1}(S^n_{1-1}(r_b))}{\max_{x} \text{vol}_{n-1}(S^n_{1-1}(r_b) \cap B^n_1(r_s, \hat{x}))} \]
\[ \geq \frac{\text{vol}_{n-1}(S^n_{1-1}(r_b))}{\text{vol}_{n-1}(S^n_{1-1}(r_s))} = \left( \frac{r_b}{r_s} \right)^{n-1} \]
and hence (16) holds.

V. RATE DISTORTION AND SPHERE COVERING IN $\ell^n_1$ WITH CONSTRAINED $\ell^n_1$-DISTORTION MEASURE

In Section IV, we have shown that the Laplacian-$\ell^n_1$ rate-distortion problem turns out to have the geometric picture of sphere covering in $\ell^n_1$, in complete analogy to the association of the Gaussian-quadratic case to sphere covering in the $\ell^2$-space. In this section, we will show that by adding an additional constraint on the $\ell^n_1$-distortion measure to break the symmetry, we can also use our geometric picture of sphere covering to explain rate distortion of an exponential source. The ultimate goal of this description will be the corresponding rate-distortion problem of a Poisson process, which can be described by the exponentially distributed inter-arrival times; see Section VI.

To motivate our choice of the constrained $\ell^n_1$-distortion measure, we make the following preliminary observations:

1) The interval description for point process realizations with a fixed number of points over duration $T$ forms a simplex (because the sum of all intervals must equal $T$), and

2) with reflections, this simplex becomes the surface (boundary) of an $\ell^n_1$-ball.

In view of these two observations, we use as a distortion measure the common $\ell^n_1$-distortion, but with the additional twist that the distortion is set to infinity if the sequences do not satisfy some specified condition. The effect of this additional constraint is to break the symmetry that was introduced in order to arrive at an $\ell^n_1$-ball from the implicitly given simplex.

A. The Exponential-$\ell^n_1$ Rate-Distortion Problem

Consider the exponential source
\[ X \sim P_X(x) = \lambda e^{-\lambda x} \Theta(x), \]
where $\Theta(\cdot)$ denotes the Heaviside function, and define the following constrained $\ell^n_1$-distortion measure.

Definition 4 (Constrained $\ell^n_1$-distortion measure):
\[ d_1(x, \hat{x}) \equiv \begin{cases} \frac{1}{n} \sum_{i=1}^{n} |x_i - \hat{x}_i| & \text{if } x_i - \hat{x}_i \geq 0 \text{ for } i \text{ odd} \\ \infty & \text{otherwise} \end{cases} \]

Remark 5: Choosing alternating signs as the condition under which the constrained $\ell^n_1$-distortion $d_1(x, \hat{x})$ is finite rather than, e.g., requiring the first half to be positive and the second half negative, makes the condition applicable still as $n \to \infty$.

Remark 6: The one-sided distortion measure for an exponential source proposed by Verdú [6], [7] is similar to Definition 4, only that the constraint is imposed such that $x_i - \hat{x}_i \geq 0$ for all $i \in \{1, 2, \ldots, n\}$.

Theorem 7: For an exponential source (19) and the constrained $\ell^n_1$-distortion measure (20), define $D \triangleq \lim_{n \to \infty} E[d_1(X, \hat{X})]$. Then the rate-distortion function is lower-bounded as
\[ R_{\text{Exponential}}(D) \geq \begin{cases} -\log(\lambda D) & \text{if } 0 < D \leq \frac{1}{\lambda} \\ 0 & \text{otherwise} \end{cases} \]

Proof: This theorem follows from Theorem 11 below.

B. Sphere Covering with Constrained $\ell^n_1$-Balls in $\mathbb{R}^n$

The exponential source specified by (19) can be viewed as the Laplacian source in (7) renormalized over the positive support $x \geq 0$. As opposed to lying almost surely on the surface of an $\ell^n_1$-ball for the Laplacian source (as $n$ tends to infinity), the sequences generated by the exponential source (19) lie instead almost surely on a simplex $\Delta^{n-1}(n/\lambda)$ in $\mathbb{R}^n$ as $n$ tends to infinity.

Since we use the constrained $\ell^n_1$-distortion measure given in (20), in the corresponding geometry of the problem, the distortion ball is also constrained: it is an orthant of the $\ell^n_1$-ball $B^n_1(r_x)$, i.e., it becomes a (nonregular) $n$-simplex, called distortion simplex.

Definition 8: For $\hat{x} \in \mathbb{R}^n$ and $r_x > 0$, we define the distortion simplex $U^r_x(\hat{x})$ as the intersection of $B^n_1(r_x, \hat{x})$ with $n$ half-spaces as follows:
\[ U^r_x(\hat{x}) \triangleq \{ x \in \mathbb{R}^n : x \in B^n_1(r_x, \hat{x}), \text{ and } x_i - \hat{x}_i \geq 0 \text{ for } i \} \]
As before, we use the shorthand notation $\mathcal{U}_n^c(r_s) \triangleq \mathcal{U}_n^c(r_s, 0)$.

Thus, we need to cover the simplex $\triangle^{n-1}(0/\lambda)$ of the exponential source (which is the first orthant of the original large $\ell_1^n$-sphere) with many small distortion simplices, see Fig. 3(b).

We use $\mathcal{H}$ to denote any hyperplane of $\mathbb{R}^n$ that is parallel to $\mathcal{H}_0$, where $\mathcal{H}_0$ is the hyperplane $x_1 + \cdots + x_n = 0$.

Proposition 9 (The maximal hyperplane section of $\mathcal{U}_n^c(1)$ is the facet $\triangle^{n-1}(1)$): Let $\mathcal{K}$ be any hyperplane in $\mathbb{R}^n$. Then

$$\max_{\mathcal{H}} \text{vol}_{\mathcal{H} \cap \mathcal{U}_n^c(1)}(\mathcal{K}) = \text{vol}_{\mathcal{K} \cap \mathcal{U}_n^c(1)}(\triangle^{n-1}(1)).$$

Proof: See Appendix B.

Corollary 10: Note that since Proposition 9 makes no assumptions about the hyperplane, it follows directly that also sections with a parallel hyperplane $\mathcal{H}$ cannot be larger than the facet $\triangle^{n-1}(1)$, i.e., $\forall n \geq 2$,

$$\max_{\mathcal{H}} \text{vol}_{\mathcal{H} \cap \mathcal{U}_n^c(1)}(\mathcal{K}) \leq \text{vol}_{\mathcal{K} \cap \mathcal{U}_n^c(1)}(\triangle^{n-1}(1)).$$

The following theorem is parallel to Theorem 3.

Theorem 11 (Asymptotic $\ell_1$-covering of a simplex): For $n \geq 2$, fix two radii $r_b > r_s > 0$ and denote by $C_n \triangleq \{x_i \in \mathbb{R}^n\}$ a collection of centers of distortion simplices $\mathcal{U}_n^c(r_s, x_i)$. Then for any $C_n$ such that

$$\triangle^{n-1}(r_b) \subseteq \bigcup_{x_i \in C_n} \mathcal{U}_n^c(r_s, x_i),$$

the following inequality holds:

$$\lim_{n \to \infty} \frac{1}{n} \log |C_n| \geq \log \left( \frac{r_b}{r_s} \right).$$

Proof: The minimal size of $C_n$ is achieved when $x_i$ are selected such that $\text{vol}_{\mathcal{K} \cap \mathcal{U}_n^c(\triangle^{n-1}(r_b)) \cap \mathcal{U}_n^c(r_s, x_i)}$ is maximized:

$$|C_n| \geq \frac{\text{vol}_{\mathcal{K} \cap \mathcal{U}_n^c(\triangle^{n-1}(r_b)) \cap \mathcal{U}_n^c(r_s, x_i)}}{\max_{\mathcal{K}} \text{vol}_{\mathcal{K} \cap \mathcal{U}_n^c(\triangle^{n-1}(r_b)) \cap \mathcal{U}_n^c(r_s, x_i)}} \geq \frac{\text{vol}_{\mathcal{K} \cap \mathcal{U}_n^c(\triangle^{n-1}(r_b)) \cap \mathcal{U}_n^c(r_s, x_i)}}{\max_{\mathcal{K}} \text{vol}_{\mathcal{K} \cap \mathcal{U}_n^c(\triangle^{n-1}(r_b)) \cap \mathcal{U}_n^c(r_s, 0)}} \geq \frac{\text{vol}_{\mathcal{K} \cap \mathcal{U}_n^c(\triangle^{n-1}(r_b)) \cap \mathcal{U}_n^c(r_s, x_i)}}{\text{vol}_{\mathcal{K} \cap \mathcal{U}_n^c(\triangle^{n-1}(r_b)) \cap \mathcal{U}_n^c(r_s, 0)}}.$$
Similarly, we have
\[ R_{\text{Poisson}}(D) = \lambda R_{\text{Exponential}}(D/\lambda) \]
and (34) follows by applying (21) of Theorem 7.

Note that we have derived a lower bound for the Poisson-$\ell_1$ rate-distortion function (Theorem 12) as a consequence of the exponential-$\ell_1$ rate-distortion problem (Theorem 7). We observe that the resulting lower bound in (34) turns out to be the rate-distortion function as when the distortion measure is the canonical queuing [8] or the point covering distortion [9], [10]. This is not a mere coincidence, and we will come back to this in a later publication where we will discuss as to why these different distortion measures yield the same rate-distortion functions.

\section*{APPENDIX A
PROOF OF PROPOSITION 2

The proof is based on a “successive slicing” argument that shows that the $(n-1)$-dimensional volume of the intersection of the small $\ell_2$-ball with the large sphere is no larger than the total surface volume of the small ball.

Let $K$ be the smallest set containing open half-spaces $K_i$ such that
\[ \bigcap_i \left( \mathbb{R}^n \setminus K_i \right) = B_{\text{reg}}(r_b). \]
(38)

Note that the size of $K$ is finite, namely $2^n$ (because $B_{\text{reg}}(r_b)$ has $2^n$ facets).

For $B_{\text{reg}}(r_b, z) \cap S_{n-1}(r_b) = \emptyset$, the claim trivially holds. Therefore we pick some $z$ such that $B_{\text{reg}}(r_b, z) \cap S_{n-1}(r_b) \neq \emptyset$.

Let $M \triangleq B_{\text{reg}}(r_b, z)$ be the small $\ell_2$-ball, and define
\[ L \triangleq \{ K \in K : K \cap M \neq \emptyset \} \subset K. \]
(39)

We now go through the following algorithm:
1) While $L \neq \emptyset$:
   a) Choose any $L \in L$.
   b) Slice off that part of $M$ that intersects with $L$:
   \[ M \leftarrow M \setminus (M \cap L). \]
   (40)
   c) Remove $L$ from $L$:
   \[ L \leftarrow L \setminus L. \]
   (41)
2) Return $M$.

Firstly, note that the algorithm will terminate in a finite number of steps because $L$ contains a finite number of elements. Secondly, observe that in each iteration, the boundary measure of $M$ is reduced. Thus, the boundary measure of the initial $M$ is greater or equal to the boundary (denoted $\partial$) measure of the returned $M$:
\[ \text{vol}_{n-1}(S_{n-1}(r_b, z)) \geq \text{vol}_{n-1}(\partial(B_{\text{reg}}(r_b, z) \cap B_{\text{reg}}(r_b))). \]
(42)

And since we also have
\[ \text{vol}_{n-1}(\partial(B_{\text{reg}}(r_b, z) \cap B_{\text{reg}}(r_b))) \geq \text{vol}_{n-1}(B_{\text{reg}}(r_b, z) \cap S_{n-1}(r_b)), \]
(43)
the claim follows.

\section*{APPENDIX B
PROOF OF PROPOSITION 9

For a regular simplex, the maximal hyperplane section is its facet [11, Sec. 5]. In this proposition, $U_{\text{reg}}^n(1)$ (an irregular $n$-simplex) is the convex hull (denoted $\text{co}()$) of a regular $(n-1)$-simplex with $0$, i.e., $U_{\text{reg}}^n(1) = \text{co}(\Delta_{n-1}(1), 0)$. Denote the origin of $\mathbb{R}^n$ by $O$, and let $P \in \mathbb{R}^n$ be the orthogonal projection of $O$ onto $\Delta_{n-1}(1)$. Then we define $O' \in \mathbb{R}^n$ such that
\[ \overrightarrow{PO'} = \sqrt{n \cdot 1} \cdot \overrightarrow{PO}, \]
(44)
and note that $\text{co}(\Delta_{n-1}(1), O')$ is a regular $n$-simplex. We denote this regular $n$-simplex by $\Delta_{\text{reg}}^n$. Clearly,
\[ U_{\text{reg}}^n(1) = \text{co}(\Delta_{n-1}(1), O) \]
(45)
\[ \subset \text{co}(\Delta_{n-1}(1), O') \]
(46)
\[ = \Delta_{\text{reg}}^n. \]
(47)

Since the maximal hyperplane section of $\Delta_{\text{reg}}^n$ is its facet, $\Delta_{n-1}(1)$ is a maximal section of $\Delta_{\text{reg}}^n$. And because $U_{\text{reg}}^n(1)$ is a subset of $\Delta_{\text{reg}}^n$ (see (46)), $\Delta_{n-1}(1)$ is also a maximal section of $U_{\text{reg}}^n(1)$, i.e., for any hyperplane $K \in \mathbb{R}^n$,
\[ \text{vol}_{n-1}(\Delta_{n-1}(1)) = \max_{K} \text{vol}_{n-1}(K \cap U_{\text{reg}}^n(1)). \]
(48)

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\section*{REFERENCES