Rate-Distortion Problems of the Poisson Process: a Group-Theoretic Approach

Hui-An Shen*, Stefan M. Moser†, and Jean-Pascal Pfister*
*University of Bern and University of Zurich ({huian.shen, jeanpascal.pfister}@unibe.ch)
†ETH Zurich and NCTU Taiwan (moser@isi.ee.ethz.ch)

***Abstract*—We study rate-distortion problems of a Poisson process using a group theoretic approach. By describing a realization of a Poisson point process with either point timings or inter-point intervals and by choosing appropriate distortion measures, we establish rate-distortion problems of a homogeneous Poisson process as ball- or sphere-covering problems for realizations of the hyperoctahedral group in $\mathbb{R}^n$. Specifically, the realizations we investigate are a hypercube and a hyperoctahedron. Thereby we unify three known rate-distortion problems of a Poisson process (with different distortion measures, but resulting in the same rate-distortion function) with the Laplacian-ℓ_1 rate-distortion problem.**

I. Introduction

A homogeneous Poisson process can be described by event (point) timings or inter-event (inter-point) intervals (compare with the left and right columns in Fig. 1). Both descriptions give rise to a group theoretic view point (conditioned on a given number of points), namely the timing description corresponds to the *symmetric group* and the interval description leads to the *reflection group*. The symmetries of these groups (in combination with properly chosen distortion measures) allow the corresponding rate-distortion problem to be expressed as a ball- or sphere-covering problem.

Concretely, in Section II we consider the symmetries in the symmetric group and its subgroup to describe the point-covering rate-distortion problem and the queueing rate-distortion problem, respectively, and show them to correspond to ℓ_∞-ball covering. In Section III we consider the symmetries in the reflection group and its subgroup to describe the exponential onesided ℓ_1-rate-distortion problem and the Laplacian ℓ_1-rate-distortion problem, respectively, and show them to correspond to ℓ_1-sphere covering.

We also introduce the concept of a *natural choice* of distortion measure that guarantees that the distortion set for a codeword has a similar shape to the source set, leading to particularly easy formulations of the ball- or sphere-covering problem and rate-distortion functions.

Finally in Section IV, we present the hyperoctahedral group which can be realized as a hypercube or a hyperoctahedron. Furthermore we show that the *symmetric group* and the *reflection group* discussed previously[1] give a construction of the hyperoctahedral group via the semidirect product, demonstrating the connections between the hyperoctahedral group and the symmetries of a Poisson process.

Notation and Definitions: We denote $[n] = \{1, 2, \dots, n\}$. For $r > 0$, $\bigcirc_1^{n-1}(r) \triangleq \bigl\{\mathbf{x} \in \mathbb{R}^n \colon \sum_{i=1}^n |x_i| = r\bigr\}$ denotes the ℓ_1-*sphere* of radius r. Its first-orthant ("hyper-surface") $(n-1)$-simplex is $\triangle^{n-1}(r) \triangleq \bigl\{\mathbf{x} \in \mathbb{R}^n \colon \sum_{i=1}^n x_i = r,\ x_i \geq 0\ \forall i \in [n]\bigr\}$.

The n-dimensional unit-cube is written as $\square^n = [0,1]^n$. By vol_n we denote the n-dimensional Lebesgue measure. Vectors are represented by bold font $\mathbf{x}$; for sets we use a calligraphic font $\mathcal{X}$; and groups are denoted by the Euler font G. The logarithm $\log(\cdot)$ is to base 2, $\mathrm{cl}(\cdot)$ denotes the closure of a set, and $\mathbb{1}\{\text{statement}\}$ represents the indicator function, which equals 1 if the statement holds true and 0 otherwise.

II. Rate Distortion and ℓ_∞-Ball Covering for the Homogeneous Poisson Process

A. The Hypercube and the Symmetric Group

Each realization of a homogeneous Poisson point process over the duration $[0, T]$ has some number of points n and can thus be described by an n-tuple $(t_1, t_2, \dots, t_n)$ where $t_1 < t_2 < \cdots < t_n$. Considering all permutations of each n-tuple and without loss of generality setting $T = 1$, the (closure of the) n-tuples and their permutations form a unit n-cube $\square^n$. We will now associate this cube with a group denoted $\mathsf{G}_n^{\mathrm{sym}}$.

Consider the symmetric group $\mathsf{S}_n = \{\sigma_1, \dots, \sigma_{n!}\}$ that includes all permutations σ_i on n objects, whose group operation is composition '$\circ$', and whose identity element $\mathsf{e}_{\mathsf{S}_n} = \sigma_1$ is the identity mapping. For any permutation $\sigma \in \mathsf{S}_n$, define the ("hyper-volume") n-simplex as

$$\mathcal{S}_\sigma \triangleq \bigl\{\mathbf{t} \in \mathbb{R}^n \colon 0 < t_{\sigma(1)} < t_{\sigma(2)} < \cdots < t_{\sigma(n)} < 1\bigr\}. \quad (1)$$

Note that this n-simplex $\mathcal{S}_\sigma$ "triangulates" the n-cube $\square^n$, and that the closure of the union of all these n-simplices forms the n-cube $\square^n$ (compare also with left upper quadrant of Fig. 1).

Definition 1: We define $\mathsf{G}_n^{\mathrm{sym}} \triangleq \{\mathcal{S}_\sigma \colon \sigma \in \mathsf{S}_n\}$ to be the associated group of the hypercube, where its group operation '$*$' is defined by means of the group operation '$\circ$' of S_n:

$$\mathcal{S}_{\sigma_j} * \mathcal{S}_{\sigma_i} \triangleq \mathcal{S}_{\sigma_j \circ \sigma_i}. \quad (2)$$

Note that the collection of ordered n-tuples describing a homogeneous Poisson process is a subgroup of $\mathsf{G}_n^{\mathrm{sym}}$. Moreover, note that $\mathsf{G}_n^{\mathrm{sym}}$ is isomorphic to S_n.

J.-P. Pfister has been supported by the Swiss National Science Foundation grants PP00P3_179060 and PP00P3_150637.

[1]The two groups that arise respectively from the timing and interval description of the Poisson process, as considered in Sections II and III.

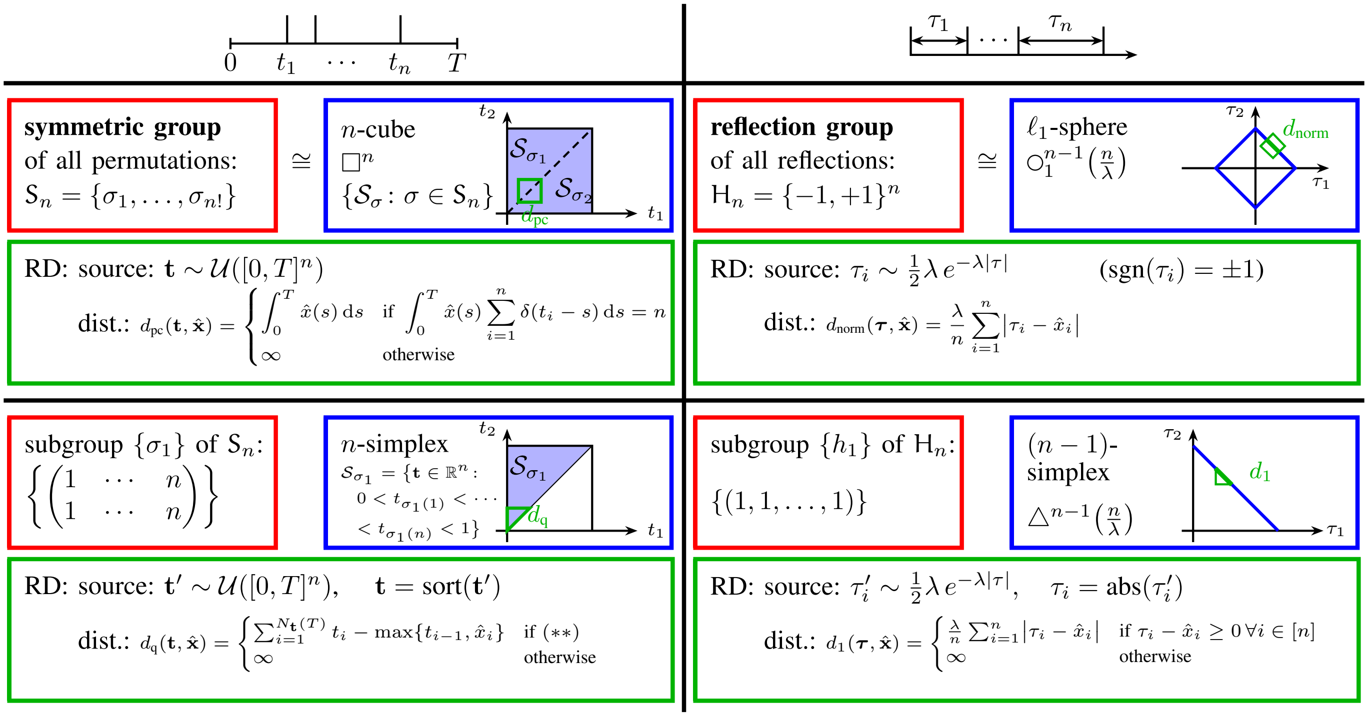

Fig. 1. Unification of four rate distortion problems. The left and right columns present the symmetries in the timing description (symmetric group) and the interval description (reflection group), respectively. The upper and lower rows present the respective group or its subgroup. Each quadrant also illustrates its own rate-distortion problem with its source set (in the blue box) and its natural distortion measure 'dist' (in the green box). In the left lower quadrant, we use Cauchy's two-line notation for permutation to denote σ_1; and the (**) condition for finite distortion is $N_{\hat{\mathbf{x}}}(T) = N_{\mathbf{t}}(T)$ and $N_{\hat{\mathbf{x}}}(s) \geq N_{\mathbf{t}}(s)\, \forall s \in [0, T]$. Note that the blue lines on the right column illustrate the region where the source is concentrated (when $n \to \infty$).

Proposition 2: The mapping $\varphi_{\text{s}} \colon \mathsf{S}_n \to \mathsf{G}_n^{\text{sym}}$, $\sigma \mapsto \mathcal{S}_\sigma$ is an isomorphism.

Proof: We have $\varphi_{\text{s}}(\sigma_j \circ \sigma_i) = \mathcal{S}_{\sigma_j \circ \sigma_i} = \mathcal{S}_{\sigma_j} * \mathcal{S}_{\sigma_i} = \varphi_{\text{s}}(\sigma_j) * \varphi_{\text{s}}(\sigma_i)$. In combination with the fact that φ_{s} is bijective, this proves that it is an isomorphism. ∎

Remark 3: Because of this isomorphism we henceforth also refer to $\mathsf{G}_n^{\text{sym}}$ as the symmetric group.

From Proposition 2 we immediately get the identity element of $\mathsf{G}_n^{\text{sym}}$:

$$\begin{aligned} \mathsf{e}_{\mathsf{G}_n^{\text{sym}}} &= \varphi_{\text{s}}(\mathsf{e}_{\mathsf{S}_n}) = \mathcal{S}_{\sigma_1} \\ &= \big\{\mathbf{t} \in \mathbb{R}^n \colon 0 < t_1 < t_2 < \cdots < t_n < 1\big\}. \end{aligned} \tag{3}$$

Thus, the subgroup $\{\mathsf{e}_{\mathsf{G}_n^{\text{sym}}}\}$ describes realizations of a homogeneous Poisson point process with n ordered points over the duration $[0, 1]$ (compare also with left lower quadrant of Fig. 1).

B. Rate-Distortion Problem on the Symmetric Group

When considering a rate-distortion problem for a certain source, sometimes there exists a *natural* choice of a distortion measure that "preserves" the geometry of the source. The most typical example is the ℓ_2-distortion measure for the Gaussian source, where the ℓ_2-distortion ball has the same fundamental shape[2] as the source ball. Based on such a geometric picture one can then use the idea of sphere- or ball-covering to derive (the converse to) the rate-distortion theorem (see, e.g., [1]).

In the following we will show how such a natural choice of distortion measure can be found for the symmetric group $\mathsf{G}_n^{\text{sym}}$ (n-cube) and for its subgroup $\{\mathsf{e}_{\mathsf{G}_n^{\text{sym}}}\}$ (n-simplex) and how they lead to two well-known rate-distortion problems of the homogeneous Poisson process, namely the point-covering distortion problem [2], [3] and the canonical queueing distortion problem [4]. We will refer to these two cases as ℓ_∞-ball covering for the homogeneous Poisson process.[3]

In the following we call the n-cube $\square^n$ or the n-simplex $\mathcal{S}_{\sigma_1}$ the *source set* and denote it by $\mathcal{T}$. Then we define the *distortion set* $\mathcal{E}_{\hat{\mathbf{x}}}(D)$ for a given codeword $\hat{\mathbf{x}} \in \hat{\mathcal{X}}$ and for an allowed distortion D (normalized by the total duration T, yielding $0 < D \leq 1$) as

$$\mathcal{E}_{\hat{\mathbf{x}}}(D) \triangleq \big\{\mathbf{t} \in \mathcal{T} \colon d(\mathbf{t}, \hat{\mathbf{x}}) \leq D\big\}. \tag{4}$$

A distortion measure $d(\cdot, \cdot)$ is said to be *natural* if the distortion set defined in (4) preserves the geometry of the corresponding source set $\mathcal{T}$ in the sense it is neccessary that

$$\text{there exists a unique } \hat{\mathbf{x}} \in \hat{\mathcal{X}} \text{ such that } \mathcal{E}_{\hat{\mathbf{x}}}(1) = \mathcal{T}. \tag{5}$$

Note that due to the normalization of timings we can set $T = 1$ without loss of generality.

[2] Recall that for large n, with very high probability the source output sequences lie in a thin sphere. Thus when refering to to the "source shape" we actually consider the geometry of the typical sequences.

[3] Other rate-distortion problems for the Poisson process that we do not consider here can be found, e.g., in [5]–[8].

1) Point-Covering Distortion: A rate-distortion codeword for the homogeneous Poisson process for the point-covering distortion is a $\{0,1\}$-valued signal $\hat{\mathbf{x}}$ on the interval $[0,1]$ (see [2], [3]). The signal $\hat{\mathbf{x}}$ partitions $[0,1]$ into a 1-valued, Lebesgue-measurable set $\mathcal{A}_{\hat{\mathbf{x}}}$ and a 0-valued set $\mathcal{A}^{\mathrm{c}}_{\hat{\mathbf{x}}}$. The *point-covering distortion measure* $d_{\mathrm{pc}}(\mathbf{t},\hat{\mathbf{x}})$ between a point process realization $\mathbf{t}$ and a codeword $\hat{\mathbf{x}}$ is the Lebesgue measure of $\mathcal{A}_{\hat{\mathbf{x}}}$, if $\mathcal{A}_{\hat{\mathbf{x}}}$ covers $\mathbf{t}$; and is infinite otherwise (see also 'dist.' in left upper quadrant of Fig. 1).

Let $\mathbf{t}$ be a Poisson point pattern of n points. Each codeword $\hat{\mathbf{x}}$ with $\mathcal{A}_{\hat{\mathbf{x}}}$ of Lebesgue measure D ($0 < D \leq 1$) gives the distortion set $\mathcal{E}_{\hat{\mathbf{x}}}(D) \subset \mathbb{R}^n$:

$$\begin{aligned}\mathcal{E}_{\hat{\mathbf{x}}}(D) &= \{\mathbf{t} \in \square^n \colon d_{\mathrm{pc}}(\mathbf{t},\hat{\mathbf{x}}) = D\} \\ &= \{\mathbf{t} \in \square^n \colon t_k \in \mathcal{A}_{\hat{\mathbf{x}}},\ \forall k \in [n]\}\end{aligned} \tag{6}$$

for $D = \mathrm{vol}_1(\mathcal{A}_{\hat{\mathbf{x}}})$. Clearly $\mathrm{vol}_n(\mathcal{E}_{\hat{\mathbf{x}}}(D)) = D^n$. The minimal number of distortion sets needed to cover the n-cube is thus $\mathrm{vol}_n(\square^n)/\mathrm{vol}_n(\mathcal{E}_{\hat{\mathbf{x}}}(D)) = 1/D^n$. This gives the minimal number of $\log(1/D)$ bits per point (i.e., per dimension).

When again including the duration T, we note that for a homogeneous Poisson process of rate λ, the expected number of points $\mathsf{E}[n] = \lambda T$. The resulting minimal average number of bits per unit time is therefore lower-bounded by $\frac{\mathsf{E}[n]}{T}\log(1/D) = \lambda\log(1/D)$, which is indeed the rate-distortion function for the Poisson process with the point-covering distortion measure [2], [3].

We have shown how the rate-distortion problem of the homogeneous Poisson process with point-covering distortion can be understood as covering an n-cube with the distortion set in (6). This cube covering perspective is similar to the converse proof given in [3], [9]. The resulting rate-distortion function shows this simple form in principle because the distortion set in (6) is matched to the source set, i.e., in other words, the point-covering distortion is a natural distortion measure for $\mathcal{T} = \square^n$ in that it satisfies (5). The geometry of the distortion set in $\mathbb{R}^n$ matches that of the symmetric group $\mathsf{G}_n^{\mathrm{sym}}$.

2) Canonical Queueing Distortion: In this section, we describe point process realizations of n points over $[0,T]$ as a tuple $\mathbf{t}$ of timings such that $t_1 < t_2 < \cdots < t_n$. Thus, when the timings are normalized by the duration T, we have $\mathbf{t} \in \mathcal{S}_{\sigma_1}$, and $\mathcal{T} = \mathcal{S}_{\sigma_1}$ is the source set (see also left lower quadrant of Fig. 1).

For the queueing rate-distortion problem, a codeword $\hat{\mathbf{x}}$ is also a point process realization over $[0,T]$ with timing description in the same ordered fashion $\hat{x}_1 < \hat{x}_2 < \hat{x}_3 < \cdots$.

Let $N_{\mathbf{P}}(\cdot)$ be the counting function on the point process $\mathbf{P}$. The queueing distortion measure is defined as [4]

$$d_{\mathrm{q}}(\mathbf{t},\hat{\mathbf{x}}) \triangleq \begin{cases} \frac{1}{T}\sum_{i=1}^{N_{\mathbf{t}}(T)} t_i - \max\{t_{i-1},\hat{x}_i\} & \text{if } N_{\hat{\mathbf{x}}}(T) = N_{\mathbf{t}}(T) \\ & \text{and } N_{\hat{\mathbf{x}}}(s) \geq N_{\mathbf{t}}(s)\, \forall s \in [0,T], \\ \infty & \text{otherwise.}\end{cases} \tag{7}$$

Without loss of generality, we continue this section by considering normalized timings for point process realizations (timings normalized by the duration T). The conditions under which $d_{\mathrm{q}}(\mathbf{t},\hat{\mathbf{x}})$ is finite can be rewritten as follows.

Proposition 4: For two (normalized) point process realizations $\mathbf{t},\hat{\mathbf{x}} \in \mathcal{S}_{\sigma_1}$ with $N_{\hat{\mathbf{x}}}(1) = N_{\mathbf{t}}(1) = n$, the following equivalence holds:

$$N_{\hat{\mathbf{x}}}(s) \geq N_{\mathbf{t}}(s)\ \forall s \in [0,1] \iff t_k \geq \hat{x}_k\ \forall k \in [n]. \tag{8}$$

Following similar arguments as in Section II-B1, we will proceed to show next how the rate distortion problem of the homogeneous Poisson process with a canonical queueing distortion measure can be understood as covering the subgroup[4] $\{\mathsf{e}_{\mathsf{G}_n^{\mathrm{sym}}}\}$ (a simplex) with a natural distortion set.

Recall Definition 1 of the symmetric group $\mathsf{G}_n^{\mathrm{sym}}$ consisting of elements that are simplices of equal n-dimensional volume, see (1). We now construct a bijection ϕ that maps each simplex $\mathcal{S}_\sigma$ (defined through a permutation on n objects) to another simplex $\tilde{\mathcal{S}}_\sigma$. The bijection ϕ is defined as follows:

$$\begin{cases} t_1 = \tilde{t}_1, \\ t_k = 1 - \sum_{i=k}^n \tilde{t}_i, \quad \text{for } k > 1,\end{cases} \tag{9}$$

i.e., we obtain the following equivalence:

$$\begin{aligned}&0 < t_{\sigma(1)} < \cdots < t_1 < \cdots < t_{\sigma(n)} < 1 \iff \\ &0 < 1 - \sum_{i=\sigma(1)}^n \tilde{t}_i < \cdots < \tilde{t}_1 < \cdots < 1 - \sum_{i=\sigma(n)}^n \tilde{t}_i < 1.\end{aligned} \tag{10}$$

Thus, ϕ maps each element in $\mathsf{G}_n^{\mathrm{sym}}$ to an element in $\tilde{\mathsf{G}}_n^{\mathrm{sym}} = \{\tilde{\mathcal{S}}_\sigma \colon \sigma \in \mathsf{S}_n\}$, and both groups consist of simplices of equal n-dimensional volume, $1/n!$. In particular, the identity element $\mathsf{e}_{\mathsf{G}_n^{\mathrm{sym}}}$ (recall (3)) maps to $\mathsf{e}_{\tilde{\mathsf{G}}_n^{\mathrm{sym}}}$:

$$\phi(\mathsf{e}_{\mathsf{G}_n^{\mathrm{sym}}}) \triangleq \mathsf{e}_{\tilde{\mathsf{G}}_n^{\mathrm{sym}}} = \left\{\tilde{\mathbf{t}} \in \mathbb{R}^n \colon \sum_{k=1}^n \tilde{t}_k < 1, \tilde{t}_k > 0\ \forall k \in [n]\right\}. \tag{11}$$

On the other hand, based on (4), the distortion set under distortion D is

$$\mathcal{E}_{\hat{\mathbf{x}}}(D) = \{\mathbf{t} \in \mathcal{S}_{\sigma_1} \colon d_{\mathrm{q}}(\mathbf{t},\hat{\mathbf{x}}) \leq D\}. \tag{12}$$

Observe that for $\hat{\mathbf{x}} = \mathbf{0}$, we have $\mathcal{E}_{\mathbf{0}}(1) = \mathcal{S}_{\sigma_1}$ and therefore, according to (5), the queueing distortion $d_{\mathrm{q}}(\cdot,\cdot)$ is a natural distortion measure for the source set $\mathcal{T} = \mathcal{S}_{\sigma_1}$.

Note that for $\hat{\mathbf{x}} = \mathbf{0}$ and arbitrary $0 < D \leq 1$

$$\mathcal{E}_{\mathbf{0}}(D) = D\mathcal{S}_{\sigma_1}, \tag{13}$$

where $D\mathcal{S}_{\sigma_1}$ denotes $\mathcal{S}_{\sigma_1}$ scaled linearly by D. On the other hand, for $\hat{\mathbf{x}} \neq \mathbf{0}$, the distortion set $\mathcal{E}_{\hat{\mathbf{x}}}(D)$ is not necessarily a simplex (this is caused by the max-function contained in the queueing distortion measure (7)), but $\hat{\mathbf{x}}$ can be chosen such that the volume of the distortion set is preserved in the following sense: For $N_{\hat{\mathbf{x}}}(1) = N_{\mathbf{t}}(1) = n$,

$$\max_{\hat{\mathbf{x}} \in \mathcal{S}_{\sigma_1}} \mathrm{vol}_n(\mathcal{E}_{\hat{\mathbf{x}}}(D)) = \mathrm{vol}_n(D\mathsf{e}_{\tilde{\mathsf{G}}_n^{\mathrm{sym}}}) = \mathrm{vol}_n(D\mathcal{S}_{\sigma_1}). \tag{14}$$

[4]We loosely say "covering a group", but actually it means covering the union of all sets that constitutes the group.

Remark 5: Note that when D is very small, for all $\mathbf{t} \in \mathcal{E}_{\hat{\mathbf{x}}}(D)$ the following holds:

$$\hat{x}_{k+1} > t_k \geq \hat{x}_k \; \forall k \in [n-1] \text{ and } 1 > t_n \geq \hat{x}_n, \tag{15}$$

and thus $d_{\mathrm{q}}(\mathbf{t}, \hat{\mathbf{x}}) = \sum_{i=1}^{N_{\mathbf{t}}(1)} (t_i - \hat{x}_i)$. Using this and Proposition 4 we get the distortion set $\mathcal{E}_{\hat{\mathbf{x}}}(D) = \hat{\mathbf{x}} + \mathrm{cl}\big(D\mathsf{e}_{\tilde{\mathsf{G}}_n^{\mathrm{sym}}}\big)$, which is an example of the distortion set shaped like a scaled version of $\mathsf{e}_{\tilde{\mathsf{G}}_n^{\mathrm{sym}}}$ in (11).

Following (14), the minimal number of distortion sets needed to cover the source n-simplex $\mathcal{T} = \mathcal{S}_{\sigma_1}$ is

$$\frac{\mathrm{vol}_n(\mathcal{S}_{\sigma_1})}{\max_{\hat{\mathbf{x}} \in \mathcal{S}_{\sigma_1}} \mathrm{vol}_n(\mathcal{E}_{\hat{\mathbf{x}}}(D))} = \frac{\mathrm{vol}_n(\mathcal{S}_{\sigma_1})}{\mathrm{vol}_n(D\mathcal{S}_{\sigma_1})} = \left(\frac{1}{D}\right)^n. \tag{16}$$

This gives again $\log(1/D)$ bits per point (per dimension) and, following the same arguments as in Section II-B1, we obtain the minimal number of bits per unit time $\lambda \log(1/D)$. This corresponds to the rate-distortion function for the Poisson process with the canonical queueing distortion measure [4].

III. Rate Distortion and ℓ_1-Sphere Covering for the Homogeneous Poisson Process

We have shown in Sections II-B1 and II-B2 that with the timing description of point process realizations, two known rate-distortion problems for the homogeneous Poisson point process (namely with point covering distortion and with the canonical queueing distortion) can be understood geometrically as minimal covering problems for the symmetric group (cube) and its subgroup (simplex), respectively. It is natural at this point to ask whether other interesting rate-distortion problems arise by considering minimal coverings of another group and its subgroup.

To that goal, recall that the inter-point interval τ of a homogeneous Poisson point process is exponentially distributed:

$$\tau \sim \lambda\, e^{-\lambda\tau}\, \mathbb{1}\{\tau \geq 0\}. \tag{17}$$

We now make the following two motivating observations:

1) If a vector of inter-point intervals describes the realization of a Poisson point process according to (17), then it lies close to a simplex $\triangle^{n-1}(n/\lambda)$ in $\mathbb{R}^n$ if n is large.
2) The number of symmetries of $\triangle^{n-1}(n/\lambda)$ in $\mathbb{R}^n$ can be increased by reflections, through which the simplex becomes the ℓ_1-sphere $\bigcirc_1^{n-1}(n/\lambda)$.

Based on these two observations and analogously to what we have shown for the symmetric group in Section II, in the rest of this section we study the reflection group and its associated rate distortion problems, namely the *Laplacian-ℓ_1* and the *exponential onesided-ℓ_1* rate-distortion problems.

A. The Hyperoctahedron and the Reflection Group

We proceed to show that $\mathsf{G}_n^{\mathrm{refl}}$ (corresponding to the ℓ_1-sphere $\bigcirc_1^{n-1}(1)$, i.e., the boundary of a regular hyperoctahedron in $\mathbb{R}^n$) is isomorphic to H_n, a reflection group that is the n-fold direct product H^n of the group $\mathsf{H} = \{+1, -1\}$ under regular multiplication '$\cdot$'. Moreover, we define for any $\mathbf{r} = (r_1, \ldots, r_n)$ and $\mathcal{A} \subseteq \mathbb{R}^n$,

$$\mathbf{r} \cdot \mathcal{A} \triangleq \big\{\mathbf{x} \in \mathbb{R}^n \colon \mathbf{a} \in \mathcal{A} \text{ and } x_k = r_k a_k \; \forall k \in [n]\big\}. \tag{18}$$

Definition 6: We define $\mathsf{G}_n^{\mathrm{refl}} \triangleq \big\{\mathsf{h} \cdot \triangle^{(n-1)}(1) \colon \mathsf{h} \in \mathsf{H}_n\big\}$ with group operation '$*$' given as follows:

$$\begin{aligned} &\big(\mathsf{h} \cdot \triangle^{(n-1)}(1)\big) * \big(\mathsf{h}' \cdot \triangle^{(n-1)}(1)\big) \\ &\quad \triangleq (\mathsf{h} \cdot \mathsf{h}') \cdot \triangle^{(n-1)}(1), \quad \mathsf{h}, \mathsf{h}' \in \mathsf{H}_n. \end{aligned} \tag{19}$$

We note that $\mathsf{G}_n^{\mathrm{refl}}$ is isomorphic to H_n.

Proposition 7: The mapping $\varphi_{\mathrm{r}} \colon \mathsf{H}_n \to \mathsf{G}_n^{\mathrm{refl}},\ \mathsf{h} \mapsto \mathsf{h} \cdot \triangle^{(n-1)}(1)$ is an isomorphism.

Proof: By its definition we obtain that φ_{r} is a bijection, and (19) establishes the homomorphism. ■

Remark 8: Because of this isomorphism we henceforth also refer to $\mathsf{G}_n^{\mathrm{refl}}$ as a reflection group.
The identity element of $\mathsf{G}_n^{\mathrm{refl}}$ is

$$\mathsf{e}_{\mathsf{G}_n^{\mathrm{refl}}} = \varphi_{\mathrm{r}}(\mathsf{e}_{\mathsf{H}_n}) = \triangle^{(n-1)}(1) \tag{20}$$

with $\mathsf{e}_{\mathsf{H}_n}$ being the identity element of H_n.

We will show in the following section that the reflection group $\mathsf{G}_n^{\mathrm{refl}}$ and its subgroup $\{\mathsf{e}_{\mathsf{G}_n^{\mathrm{refl}}}\}$ with their respective natural distortion measures yield the *Laplacian-ℓ_1* and the *exponential onesided-ℓ_1* rate-distortion problem.

B. Rate-Distortion Problem on the Reflection Group

Similarly to the discussion for the symmetric group in Section II-B, we now consider the rate-distortion and minimal covering problem on the reflection group and its subgroup: $\mathsf{G}_n^{\mathrm{refl}}$ (ℓ_1-sphere $\bigcirc_1^{n-1}(1)$) and $\{\mathsf{e}_{\mathsf{G}_n^{\mathrm{refl}}}\}$ (simplex $\triangle^{(n-1)}(1)$).

Recall from Observation 1) of this section that the inter-point interval realizations generated by (17) lie almost surely in the thin shell around $\triangle^{n-1}(n/\lambda)$ (for a sufficiently large number of intervals); compare also with the schematic in right lower quadrant in Fig. 1. Furthermore, we implement Observation 2) by labeling each inter-point interval with -1 or 1 equiprobably. This labeling creates a new source τ_{s} of signed inter-point intervals that has a Laplacian distribution:

$$\tau_{\mathrm{s}} \sim \frac{\lambda}{2}\, e^{-\lambda|\tau_{\mathrm{s}}|}, \tag{21}$$

whose realizations of length-n sequences lie almost surely in the thin shell around the ℓ_1-sphere $\bigcirc_1^{n-1}(n/\lambda)$ (compare also with the schematic in right upper quadrant in Fig. 1). We use again the notions of the source set $\mathcal{T}$ and natural distortion measure introduced in Section II-B, and we consider two source sets $\mathcal{T} = \bigcirc_1^{n-1}(n/\lambda)$ or $\mathcal{T} = \triangle^{(n-1)}(n/\lambda)$, with their respective natural distortion measures. We refer to these two cases as ℓ_1-sphere covering.

Again, using the same ideas based on the geometric picture of source set and distortion set, one can derive the rate-distortion functions for these two rate-distortion problems, see for example [1]. In the following we will only briefly summarize the results and omit their geometric derivations.

1) Laplacian-ℓ_1 Rate-Distortion Problem: The *normalized ℓ_1-distortion measure* is defined as

$$d_{\mathrm{norm}}(\mathbf{x}, \hat{\mathbf{x}}) \triangleq \frac{\lambda}{n} \sum_{i=1}^{n} |x_i - \hat{x}_i|, \tag{22}$$

where λ is the parameter of the Laplacian source in (21). It is easy to verify that the normalized ℓ_1-distortion measure is a natural distortion measure for $\mathcal{T} = \bigcirc_1^{n-1}(n/\lambda)$. The following lemma now follows from [10, Lemma 6].

Lemma 9: For a Laplacian source (21) and the normalized ℓ_1-distortion measure (22), the rate distortion function is

$$R_{\text{Laplacian}}(D) = \log\left(\frac{1}{D}\right) \mathbb{1}\{0 < D \leq 1\}. \tag{23}$$

2) Exponential Onesided-ℓ_1 Rate-Distortion Problem: The *normalized onesided ℓ_1-distortion measure* is defined as

$$d_1(\mathbf{x}, \hat{\mathbf{x}}) \triangleq \begin{cases} \frac{\lambda}{n}\sum_{i=1}^{n} |x_i - \hat{x}_i| & \text{if } x_i - \hat{x}_i \geq 0 \ \forall i \in [n], \\ \infty & \text{otherwise,} \end{cases} \tag{24}$$

where λ is the parameter of the exponential source in (17). Again, one can verify that $d_1(\cdot,\cdot)$ is a natural distortion measure for $\mathcal{T} = \triangle^{(n-1)}(n/\lambda)$. The following lemma now follows directly from [10, Lemma 2].

Lemma 10: For an exponential source (17) and the normalized onesided ℓ_1-distortion measure (24), the rate-distortion function is

$$R_{\text{Exponential}}(D) = \log\left(\frac{1}{D}\right) \mathbb{1}\{0 < D \leq 1\}. \tag{25}$$

Note that when viewing n as the number of points in a point process realization, the rate-distortion functions in Lemmas 9 and 10 are the same function, measured in bits per symbol (per point). This gives $\log(1/D)$ bits per point just as the results in Sections II-B1 and II-B2. Therefore, when considering the rate-distortion problem under the inter-point interval description of a homogeneous Poisson process, we can follow similar arguments as in Section II-B1 to get the minimal number of bits per unit time $\lambda \log(1/D)$.

It is not a coincidence that all four rate-distortion functions in Sections II-B1, II-B2, III-B1, III-B2 are the same. The reason is that they all have their own natural distortion sets matched to their source sets, i.e., they all satisfy the criterion (5).

To this point, we have presented the rate-distortion problems of the Poisson process as ℓ_∞-ball covering in Section II and ℓ_1-sphere covering in Section III. One may wonder why it exactly is ℓ_∞ and ℓ_1. We attempt to answer this question in the following section by exploring the *hyperoctahedral group*.

IV. The Hyperoctahedral Group

In this section we use the following standard group-theoretic definition for the *semidirect product*.

Definition 11 (Internal Semidirect Product): Let H_1 and H_2 be subgroups of G equipped with the group operation '$\cdot$' and with the identity element e_G. We say that G is the *internal semidirect product* of H_1 by H_2, denoted $\mathsf{G} = \mathsf{H}_1 \rtimes \mathsf{H}_2$, if

- H_1 is a normal subgroup of G, i.e., $\mathsf{g} \cdot \mathsf{H}_1 = \mathsf{H}_1 \cdot \mathsf{g}$ for all $\mathsf{g} \in \mathsf{G}$;
- $\mathsf{H}_1 \cap \mathsf{H}_2 = \{\mathsf{e}_\mathsf{G}\}$;
- $\mathsf{G} = \mathsf{H}_1 \cdot \mathsf{H}_2$.

The *hyperoctahedral* group (see e.g. [11]) O_n describes the symmetries of an n-dimensional hypercube or of an n-dimensional regular hyperoctahedron (cross-polytope).

Claim 12: The n-cube and the regular n-dimensional hyperoctahedron have the same group of symmetries O_n.
Thus, both the n-cube and the regular n-hyperoctahedron are realizations of the group of symmetries O_n. The *order* $|\mathsf{O}_n|$ of the hyperoctahedral group O_n is $|\mathsf{O}_n| = 2^n n!$. For example in three dimensions, O_3 can be understood as a composition of (rigid-body) rotation and mirroring, which gives $|\mathsf{O}_3| = 2 \cdot 24 = 48$.

The n-cube and the n-hyperoctahedron do not only have the same group of symmetries, they are actually geometric duals: replacing the vertices of one by $(n-1)$-dimensional faces results in the other and vice-versa.

Now define an n-cube by its graph $Q_n = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{0,1\}^n$ is the set of vertices and $\mathcal{E}$ is the set of edges. Using this we can alternatively define the *hyperoctahedral* group as the automorphism group $\mathsf{Aut}(Q_n)$ of Q_n [12, Lecture 3]. Following standard notation, let $\mathsf{Z}_2 \triangleq \{0,1\}$ be a group equipped with modulo-2 addition, and let Z_2^n be its n-fold direct product. It is known that $\mathsf{Aut}(Q_n)$ is isomorphic (denoted by $\cong$) to the internal semidirect product of Z_2^n by the symmetric group S_n [13], [12, Lecture 3]: $\mathsf{Aut}(Q_n) \cong \mathsf{Z}_2^n \rtimes \mathsf{S}_n$. Clearly, $\mathsf{Z}_2^n \cong \mathsf{H}_n$. And since by Propositions 2 and 7 we have $\mathsf{S}_n \cong \mathsf{G}_n^{\text{sym}}$ and $\mathsf{H}_n \cong \mathsf{G}_n^{\text{refl}}$, we obtain the following result.

Theorem 13:

$$\mathsf{O}_n \triangleq \mathsf{Aut}(Q_n) \cong \mathsf{G}_n^{\text{refl}} \rtimes \mathsf{G}_n^{\text{sym}}. \tag{26}$$

By the third condition in Definition 11 for the internal semidirect product, we can understand Theorem 13 intuitively as the construction of O_n by two of its subgroups: the reflection subgroup $\mathsf{G}_n^{\text{refl}}$ and the symmetric subgroup $\mathsf{G}_n^{\text{sym}}$. Recall from the definitions of $\mathsf{G}_n^{\text{sym}}$ and $\mathsf{G}_n^{\text{refl}}$ (Definitions 1 and 6) that they are associated with a hypercube and a regular hyperoctahedron respectively. Referring back to the graphical summary of Sections II and III in Fig. 1, we conclude from this section that the hyperoctahedral group unifies the two columns of Fig. 1, demonstrating the symmetries of a Poisson process.

Acknowledgments

We thank the reviewers for very helpful comments and further references. Especially, we are grateful to the reviewer who pointed out issues related to (5).

References

[1] H.-A. Shen, S. M. Moser, and J.-P. Pfister, "Sphere covering for Poisson processes," in *Proc. 2020 IEEE Inf. Theory Workshop*, Riva del Garda, Italy, Apr. 11–15, 2021, pp. 181–185.

[2] A. Lapidoth, A. Malär, and L. Wang, "Covering point patterns," in *Proc. IEEE Int. Symp. Inf. Theory*, St. Petersburg, Russia, Jul. 31 – Aug. 5, 2011, pp. 51–55.

[3] ——, "Covering point patterns," *IEEE Trans. Inf. Theory*, vol. 61, no. 9, pp. 4521–4533, Sept. 2011.

[4] T. P. Coleman, N. Kiyavash, and V. G. Subramanian, "The rate-distortion function of a Poisson process with a queueing distortion measure," in *Proc. Data Compression Conf.*, Snowbird, UT, USA, Mar. 25–27, 2008, pp. 63–72.

[5] I. Rubin, "Rate-distortion functions for nonhomogeneous Poisson processes," *IEEE Trans. Inf. Theory*, vol. 20, no. 5, pp. 669–672, Sept. 1974.

[6] ——, "Information rates and data-compression schemes for Poisson processes," *IEEE Trans. Inf. Theory*, vol. 20, no. 2, pp. 200–210, Mar. 1974.

[7] ——, "Information rates for Poisson sequences," *IEEE Trans. Inf. Theory*, vol. 19, no. 3, pp. 283–294, May 1973.

[8] N. V. Shende and A. B. Wagner, "Functional covering of point processes," in *Proc. IEEE Int. Symp. Inf. Theory*, Paris, France, Jul. 7–12, 2019, pp. 2039–2043.

[9] A. Mazumdar and L. Wang, "Covering arbitrary point patterns," in *Proc. 50th Allerton Conf. Commun., Control Comput.*, Monticello, IL, USA, Oct. 1–5, 2012, pp. 2075–2080.

[10] H. Si, O. O. Koyluoglu, and S. Vishwanath, "Lossy compression of exponential and Laplacian sources using expansion coding," in *Proc. IEEE Int. Symp. Inf. Theory*, Honolulu, HI, USA, Jun. 29 – Jul. 4, 2014, pp. 3052–3056.

[11] M. Baake, "Structure and representations of the hyperoctahedral group," *J. Math. Physics*, vol. 25, no. 11, pp. 3171–3182, Nov. 1984.

[12] P. Gregor, *Hypercube Structures (Lecture)*, Charles University, Prague, Czech Republic, 2020. [Online]. Available: https://ktiml.mff.cuni.cz/ gregor/hypercube/hypercube-course.htm

[13] F. Harary, "The automorphism group of a hypercube," *J. Univ. Comp. Sc.*, vol. 6, no. 1, pp. 136–138, Jan. 2000.