The Geometry of Uncoded Transmission for Symmetric Continuous Log-Concave Distributions

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Abstract—**We present a geometric picture for optimal single-letter uncoded transmission for *source-channel duals*, where the source and distortion measure are dual to the channel and cost function. In particular, we investigate an additive noise channel with the conditional channel distribution and capacity-achieving input distribution both being symmetric, continuous log-concave densities. We show that under these assumptions, a Gaussian source transmitted over an additive Gaussian channel is the only possible choice for optimal single-letter uncoded transmission. We explain the uniqueness of Gaussian uncoded transmission through a *homothetic property* for the channel input and output typical sets, and illustrate the geometry of single-letter uncoded transmission as opposed to communication based on the classical source-channel separation principle.**

I. Introduction

As proven in Shannon's source-channel separation theorem, it is possible to design information-theoretically optimal systems that achieve both the rate-distortion function of the source and the capacity-cost function of the channel. Unfortunately, in Shannon's approach, the optimal system employs a code with high complexity and infinite delay.

In the context of biological plausibility, [1] proposed optimal *almost* code-free information transmission, for which it is required that both encoder and decoder only perform linear scaling (or are identity functions). For brevity, we will refer to such schemes as *uncoded transmission*.

Uncoded transmission was studied systematically by means of *probabilistic matching* in [2], and it follows that there exist infinite quadruples of source, encoder, channel, and decoder that are optimal over single-letter transmission, i.e., with codewords of length 1 that cause no delay. Such single-letter transmission works when both the distortion measure and the cost function are probabilistically matched to the said quadruple.

Two well-known examples of uncoded transmission are a binary source over a binary symmetric channel, using Hamming distortion (see [1, Ex. 1, Sec. 4.1]), and a Gaussian source over the Gaussian channel with squared-error distortion and second-moment constraint (see [1, Ex. 2, Sec. 4.2]). Note that both cases are information-theoretic *source-channel duals*.[1] This duality permits the aforementioned encoder and decoder to be functionally equivalent (see, e.g., [3]).

It is a curious fact that, for continuous sources and channels, no other source-channel duals for optimal single-letter uncoded transmission have been discovered other than the aforementioned Gaussian case. In the following we are going to partially explain why this is the case. We develop, under mild assumptions, the geometry of optimal uncoded transmission over continuous additive noise channels with its dual source. The assumptions we take are that a) both the channel and the capacity-achieving input distribution are symmetric continuous log-concave distributions, and b) that the aforementioned encoder and decoder are identical linear functions acting on a single letter. The main "geometric reasoning" for restricting to log-concave distributions is to make use of concentration inequalities for log-concave random variables, and to associate its density function with a convex body. The restriction for the encoder and decoder being identical functions arises from the functional equivalence between the source encoder and channel decoder in dual problems of source and channel coding, as introduced in the previous paragraph.

Our main theorems illustrate through a geometric perspective that the well-known Gaussian uncoded transmission is indeed the unique solution for source-channel duals where the latter is the family of all symmetric continuous log-concave additive noise channels. So, e.g., the corresponding situation of an Laplacian source and an ℓ_1-distortion measure that is transmitted over an additive Laplacian noise channel does *not* allow for uncoded transmission. In fact, any generalization of the ℓ_2 case (Gaussian case) to a general ℓ_p-normed case (for $p \neq 2$) does not work.

We show that uncoded transmission naturally arises when the input and output distributions of the channel yield not only "linearly equivalent[2]" but also "*homothetic*[3] typical sets", which we call the *homothetic property*. When the distortion measure is associated with a norm generated by an inner product, this allows for linear single-letter codes that are optimal.

This work was supported by the Swiss National Science Foundation grants PP00P3_179060 (J.-P.P.) and 31003A_175644 (H.-A.S.).

[1]Duality here is in the sense of source coding and channel coding. Therefore, strictly speaking, the duality is between the pairs "source and distortion measure" and "channel and cost function". See Section III for more details.

[2]Two sets $\mathcal{T}_1, \mathcal{T}_2$ are linearly equivalent when there is a nonsingular linear transformation ϕ such that $\phi(\mathcal{T}_1) = \mathcal{T}_2$.

[3]Here *homothetic* is based on the idea of *homothets* of a convex body, as we will define in Section II.

II. Definitions and Notation

We use capitalized Roman alphabets, e.g. X, to denote random variables (RV), with the exception of "V" that is exclusively reserved to denote a real vector space. We write $X \perp\!\!\!\perp Z$ to denote X and Z being independent RVs; f_X denotes the probability density function of the RV X. A density over $\mathbb{R}$ is said to be *symmetric* if $f_X(x) = f_X(-x)$, $\forall x \in \mathbb{R}$. For $k \in \mathbb{N}$, we define $[k] \triangleq \{1, 2, \ldots, k\}$.

We denote $\{\mathbf{e}_i\}_{i\in[n]}$ to be the standard orthonormal basis of an n-dimensional real vector space V. Bold font denotes a tuple or a vector, and $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $x_i \in \mathbb{R}$, refers to either a point $\mathbf{x} \in \mathbb{R}^n$ or $\mathbf{x} = \sum_{i\in[n]} x_i \, \mathbf{e}_i \in V$. In this paper, $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{x} \in V$ are used interchangeably.

We use $\mathrm{conv}(\cdot)$ to denote the convex hull. Sets and convex bodies in $\mathbb{R}^n$ are denoted with calligraphic font, e.g. $\mathcal{K}$. A convex body $\mathcal{K} \subset \mathbb{R}^n$ is a compact convex set with nonempty interior. A *homothet* of a convex body $\mathcal{K} \subset \mathbb{R}^n$ is any set with the form $\mathbf{x} + \lambda\mathcal{K} = \{\mathbf{x} + \lambda\mathbf{t} \colon \mathbf{t} \in \mathcal{K}\}$ for some $\mathbf{x} \in \mathbb{R}^n$ and nonzero $\lambda \in \mathbb{R}$.

For an o-symmetric (i.e., centrally symmetric with regard to the origin o) convex body $\mathcal{K}$ in $\mathbb{R}^n$, we use $\|\cdot\|_{\mathcal{K}}$ to denote the norm defined by (the gauge of) $\mathcal{K}$, i.e., for any $\mathbf{x} \in \mathbb{R}^n$,

$$\|\mathbf{x}\|_{\mathcal{K}} = \inf\{\lambda > 0 \colon \mathbf{x} \in \lambda\mathcal{K}\}. \tag{1}$$

In this paper, normed spaces are always real vector spaces.

III. Duality between Source and Channel Coding

This section defines *source-channel duals* as first encountered in Section I, in terms of a dual source coding problem to a channel coding problem. This is based on the classical information-theoretic duality (see, e.g., [3, Sec. III]).

Definition 1: For a channel coding problem with input variable X, output variable Y, conditional channel distribution $P_{Y|X}$, and capacity-achieving input distribution P_X^* inducing the channel output marginal $\bar{P}_Y$, the *dual source coding problem* is the rate-distortion problem over the source variable Y with distribution $\bar{P}_Y$, reconstruction variable X, and the single-letter distortion measure $d\colon \mathcal{Y} \times \mathcal{X} \to \mathbb{R}_0^+$ taking the form

$$d(y, x) = -c_0 \log P_{Y|X}(y|x) + d_0(y), \tag{2}$$

for arbitrary $c_0 > 0$ and $d_0(\cdot)$.
The following theorem is a corollary to [2, Th. 6].

Theorem 2: For the channel coding problem stated in Definition 1, define the induced *backward channel* as

$$\bar{P}_{X|Y}(x|y) = \frac{P_X^*(x) P_{Y|X}(y|x)}{\bar{P}_Y(y)}. \tag{3}$$

Then for the dual source coding problem according to Definition 1, we have

$$\bar{P}_{X|Y}(x|y) = \underset{\substack{Q_{X|Y} \\ \mathsf{E}[d(Y,X)] \le D}}{\operatorname{argmin}} \; \mathrm{I}\big(\bar{P}_Y, Q_{X|Y}\big), \tag{4}$$

$$\mathrm{I}\big(P_X^*, P_{Y|X}\big) = \min_{\substack{Q_{X|Y} \\ \mathsf{E}[d(Y,X)] \le D}} \; \mathrm{I}\big(\bar{P}_Y, Q_{X|Y}\big), \tag{5}$$

where the distortion $D = \mathsf{E}_{Q^*}[d(Y, X)]$ with $Q^*(y, x) = P_X^*(x) P_{Y|X}(y|x)$.

IV. Uncoded Transmission in Normed Spaces

A. Preliminaries Regarding Normed and Inner Product Spaces

Definition 3 (Real normed linear spaces): A real normed linear space $(V, \|\cdot\|)$ is a real vector space V with a function (called *norm*) $\|\cdot\| \colon V \to \mathbb{R}$, $\mathbf{x} \mapsto \|\mathbf{x}\|$ satisfying the following properties:

(1) $\|\mathbf{x}\| \ge 0$, $\forall \mathbf{x} \in V$;
(2) $\|\mathbf{x}\| = 0$ if, and only if, $\mathbf{x} = \mathbf{0}$;
(3) $\|\lambda\mathbf{x}\| = |\lambda| \, \|\mathbf{x}\|$, $\forall \lambda \in \mathbb{R}$, $\forall \mathbf{x} \in V$;
(4) $\|\mathbf{x} + \mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$, $\forall \mathbf{x}, \mathbf{y} \in V$.

Definition 4 (Real inner product spaces): A real inner product space $(V, \langle\cdot,\cdot\rangle)$ is a real vector space with a function (called *inner product*) $\langle\cdot,\cdot\rangle \colon V \times V \to \mathbb{R}$, $(\mathbf{x}, \mathbf{y}) \mapsto \langle\mathbf{x}, \mathbf{y}\rangle$ satisfying the following properties:

(1) $\langle\mathbf{x}, \mathbf{x}\rangle \ge 0$, $\forall \mathbf{x} \in V$;
(2) $\langle\mathbf{x}, \mathbf{x}\rangle = 0$ if, and only if, $\mathbf{x} = \mathbf{0}$;
(3) $\langle\mathbf{x}, \mathbf{y}\rangle = \langle\mathbf{y}, \mathbf{x}\rangle$, $\forall \mathbf{x}, \mathbf{y} \in V$;
(4) $\langle\lambda\mathbf{x}, \mathbf{y}\rangle = \lambda\langle\mathbf{x}, \mathbf{y}\rangle$, $\forall \lambda \in \mathbb{R}$, $\forall \mathbf{x}, \mathbf{y} \in V$;
(5) $\langle\mathbf{x} + \mathbf{z}, \mathbf{y}\rangle = \langle\mathbf{x}, \mathbf{y}\rangle + \langle\mathbf{z}, \mathbf{y}\rangle$, $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in V$.

Theorem 5: For any real inner product space $(V, \langle\cdot,\cdot\rangle)$, define

$$\|\mathbf{x}\| \triangleq \sqrt{\langle\mathbf{x}, \mathbf{x}\rangle}, \quad \forall \mathbf{x} \in V. \tag{6}$$

Then $(V, \|\cdot\|)$ is a real normed linear space.

Definition 6: Let $\|\cdot\|$ be a norm on V. We say "$(V, \|\cdot\|)$ is an inner product space" when there exists an inner product $\langle\cdot,\cdot\rangle$ such that (6) is satisfied.

B. Preliminaries Regarding Log-concavity

Definition 7 (Log-concave functions): A function $f\colon \mathbb{R}^n \to [0, +\infty)$ is said to be *log-concave* if it has the form

$$f = e^{-g} \tag{7}$$

where $g\colon \mathbb{R}^n \to (-\infty, \infty]$ is a convex function.

Definition 8: We say "X is a log-concave random variable" when f_X is a log-concave function.

Proposition 9 (Log-concavity preserved over convolution [4]): If X, Z are independent log-concave random variables, then $X + Z$ is also log-concave.

C. Main Theorem 1

Definition 10: The differential entropy of a probability density function f is given as

$$\mathrm{h}(f) \triangleq -\int_{-\infty}^{\infty} f(s) \log f(s) \, \mathrm{d}s. \tag{8}$$

Definition 11: Let Ψ be the family of continuous symmetric log-concave probability density functions over $\mathbb{R}$, and Ω be the collection of compact convex sets in $\mathbb{R}^n$ with the collection of their boundaries $\partial\Omega$. For $f \in \Psi$, define $\Phi_n \colon \Psi \to \partial\Omega$, $f \mapsto \Phi_n(f)$ as follows:

$$\Phi_n(f) \triangleq \left\{\mathbf{x} \in \mathbb{R}^n \colon -\frac{1}{n}\sum_{i=1}^{n} \log f(x_i) = \mathrm{h}(f)\right\}. \tag{9}$$

Theorem 12 (Main Theorem 1): For any $n \geq 2$, let V be an n-dimensional real vector space, and let $\mathcal{B}_n(f)$ denote the convex hull of $\Phi_n(f)$. For a log-concave random variable $Z \in \mathbb{R}$ with a symmetric, continuous density f_Z, define the family $\mathcal{F}$ of log-concave random variables $X \in \mathbb{R}$ where $X \perp\!\!\!\perp Z$ and f_X is symmetric, continuous and satisfies

$$t\Phi_n(f_X) = \Phi_n(f_Z), \qquad \text{for some } t > 0. \tag{10}$$

Then the following three conditions are equivalent:

(i) For all $X \in \mathcal{F}$, $(V, \|\cdot\|_{\mathcal{K}})$ is an inner product space for $\mathcal{K} = \mathcal{B}_n(f_X)$.

(ii) For all $X \in \mathcal{F}$, there exists some $\alpha_t > 0$ which only depends on t, such that

$$\Phi_n(f_{X+Z}) = \alpha_t \Phi_n(f_X). \tag{11}$$

(iii) For all $X \in \mathcal{F}$ and $\mathcal{K} = \mathcal{B}_n(f_X)$, there exists some $\alpha_t > 0$ which only depends on t, such that for any pair of $\mathbf{y} \in \Phi_n(f_{X+Z})$, $\mathbf{x} \in \Phi_n(f_X)$,

$$\|\mathbf{y} - \mathbf{x}\|_{\mathcal{K}} = \left\| \frac{1}{\alpha_t}\mathbf{y} - \alpha_t \mathbf{x} \right\|_{\mathcal{K}} \tag{12}$$

and $\alpha_t \neq 1$.

Proof: See Appendices A and B. ∎

Remark 13: Property (ii) describes the "homothetic property" mentioned in Section I, and Property (iii) describes the geometry of uncoded transmission with a linear single-letter code over the source-channel duals. We will discuss these conditions further in Sections IV-D and IV-E.

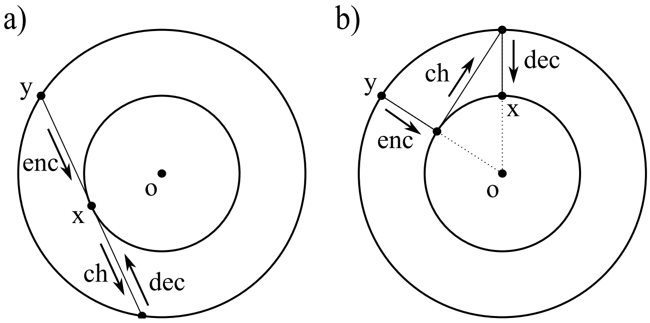

Fig. 1. A schematic comparison of a) optimal infinite-letter coded transmission, and b) optimal single-letter (scaled) uncoded transmission. Both source-channel communication systems consist sequentially of an encoder (enc), a channel (ch), and a decoder (dec). In a), the encoder and decoder are jointly typical encoder/decoders that act on infinite-length sequences. In b), the encoder and decoder can act on single symbols by scaling due to the homothetic property.

D. Main Theorem 2: Strengthening Theorem 12

In this section, we relax two assumptions in Theorem 12 to obtain Theorem 14: we do not require that f_X and f_Z yield linearly equivalent sets by Φ_n (i.e., assumption (10) is relaxed), and we also do not require in the third condition that $\mathcal{K}$ is a convex body associated with f_X (i.e., $\mathcal{K} = \mathcal{B}_n(f_X)$ in Condition (iii) of Theorem 12 is relaxed).

Theorem 14 (Main Theorem 2): For any $n \geq 2$, let V be an n-dimensional real vector space. Let $Z \in \mathbb{R}$ be a log-concave random variable with a symmetric, continuous density f_Z. Define the family $\mathcal{F}$ of log-concave random variables over $\mathbb{R}$ with symmetric, continuous densities satisfying for any $X_i, X_j \in \mathcal{F}$,

$$u\Phi_n(f_{X_i}) = \Phi_n(f_{X_j}), \qquad \text{for some } u > 0; \tag{13}$$

and $X \perp\!\!\!\perp Z$ for any $X \in \mathcal{F}$. Then the following three conditions are equivalent:

(I) The RV Z and all RVs in the family $\mathcal{F}$ are Gaussian.

(II) For all $X_i \in \mathcal{F}$, there exists some $\alpha_i > 0$, such that

$$\Phi_n(f_{X+Z}) = \alpha_i \Phi_n(f_X). \tag{14}$$

(III) For all $X_i \in \mathcal{F}$, and for any norm $\|\cdot\|$ generated by an inner product on V, there exists some $\alpha_i > 0$, such that for any pair of $\mathbf{y} \in \Phi_n(f_{X+Z})$, $\mathbf{x} \in \Phi_n(f_X)$,

$$\|\mathbf{y} - \mathbf{x}\| = \left\| \frac{1}{\alpha_i}\mathbf{y} - \alpha_i \mathbf{x} \right\| \tag{15}$$

and $\alpha_i \neq 1$.

Proof: We omit the proof of Theorem 14 for it is similar to that of Theorem 12. We only remark that for showing (II) $\Longrightarrow$ (I), we make use of the following proposition.

Proposition 15: For $X \perp\!\!\!\perp Z$, if f_X and f_{X+Z} are both zero-mean Gaussian distributions, then f_Z is also Gaussian. ∎

The graphical representation of Gaussian uncoded transmission, which uniquely satisfies Properties (i)–(iii) and (I)–(III) in Theorem 12 and 14, is given in Fig. 1b).

E. Interpreting the Main Theorems for the Geometry of Uncoded Transmission

To see how the main theorems, Theorems 12 and 14, explain the geometry of uncoded transmission, we first introduce the following theorem.

Theorem 16 ([5, Th. 2]): Let $\mathbf{X}^{(n)}$ be a random vector in $\mathbb{R}^n$ with log-concave density f. Then for any $0 \leq t \leq 2$,

$$\Pr\left[\frac{1}{n}\left|\log f(\mathbf{X}^{(n)}) - \mathsf{E}\left[\log f(\mathbf{X}^{(n)})\right]\right| \geq t\right] \leq 4\, e^{-ct^2 n} \tag{16}$$

where $c \geq \frac{1}{16}$.

Intuitively, Theorem 16 describes how the random vector realizations concentrate near a 'thin shell' as $n \to \infty$. In the special case when each element of the random vector is generated IID from f_X, a log-concave density over $\mathbb{R}$, this 'thin shell' lies around $\Phi_n(f_X)$ (see Definition 11) and can be understood as the *typical set* of X for some n sufficiently large. Note that in the main theorems, both X and $X + Z$ are log-concave random variables and thus Theorem 16 applies. In this context, loosely speaking, Property (ii) and (II) state that the typical sets of $X + Z$ and X are linearly equivalent as $n \to \infty$. We call this the "homothetic property", and depict it in Fig. 1 with the outer and inner sphere, respectively, representing the typical sets of $X + Z$ and X.

Let the channel coding problem be on input variable X with capacity-achieving distribution P_X^*, output variable $Y = X + Z$ with additive channel noise Z. Then using Definition 1 we have the dual source coding problem on source variable Y and reconstruction variable X. In this case we have the following communication system on a source-channel dual with length-n letters:

$$\mathbf{Y}^{(n)} \xrightarrow{\text{enc}} \mathbf{x}^{(n)} \xrightarrow{\text{ch}} \mathbf{Y}^{(n)} \xrightarrow{\text{dec}} \mathbf{X}^{(n)}, \tag{17}$$

where 'enc' stands for 'encoder', 'ch' for 'channel', and 'dec' for 'decoder'. As n becomes sufficiently large, we can understand the system in (17) as acting on sequences in the typical set for arbitrary $\epsilon > 0$:

$$\mathcal{T}_\epsilon^{(n)}(Y) \xrightarrow{\text{enc}} \mathcal{T}_\epsilon^{(n)}(X) \xrightarrow{\text{ch}} \mathcal{T}_\epsilon^{(n)}(Y) \xrightarrow{\text{dec}} \mathcal{T}_\epsilon^{(n)}(X). \tag{18}$$

The classical scenario for optimal source-channel dual communication system is shown in Fig. 1a), where the encoder is the *jointly typical encoder*, and the decoder is the *jointly typical decoder*, which 'undoes' the channel (i.e., we have reliable transmission). However, both the encoder and decoder in the classical scenario need to act on vectors of length n. In contrast, the homothetic property presented in the two main theorems allows for uncoded transmission, as shown in Fig. 1b). This means that the encoder can perform a linear scaling isotropically independent of the position of the source sequence $\mathbf{y}$ on the outer sphere. This makes it possible for the encoder to act on a single-letter allowing for uncoded transmission. Since the decoder is equivalent to the encoder due to the imposed source-channel duality, the same reasoning also applies to the decoder.

Appendix A
"Homothetic Property" of Uncoded Transmission and the Inner Product Space

A. Characterizations of Inner Product Spaces

Since an inner product space is a normed linear space with extra structure (see also Theorem 5 and Definition 6), there are properties that characterize inner product spaces which help us determine whether the norm arises from an inner product in the sense of (6). One such characterization is the well-known *Parallelogram Law*[4] [6]. Another such characterization is presented as follows.

Theorem 17 ([7, Th. 2.2.]): Let $X = (V, \|\cdot\|)$ be a 2-dimensional real normed linear space and let its unit sphere be $\mathcal{S}_X \triangleq \{\mathbf{x} \in \mathbb{R}^2 \colon \|\mathbf{x}\| = 1\}$. Then X is an inner product space if, and only if, $\mathcal{S}_X$ is an ellipse.

In the following section, we use the characterization presented in Theorem 17 to give a proof for (i) $\Longrightarrow$ (ii) of Theorem 12. We isolate this particular part of the proof of Theorem 12 to illustrate the underlying geometric property (as in Theorem 17) of an inner product space, before we present the rest of the proof for Theorem 12 in Appendix B.

[4] A normed linear space $(V, \|\cdot\|)$ is an inner product space if, and only if,

$$2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2 = \|\mathbf{x}-\mathbf{y}\|^2 + \|\mathbf{x}+\mathbf{y}\|^2, \quad \forall \mathbf{x}, \mathbf{y} \in V.$$

B. Proof for (i) $\Longrightarrow$ (ii) in Theorem 12

1) For $n = 2$, we apply Theorem 17 to see that the unit sphere $\Phi_2(f_X)$ is an ellipse. Taking $n = 2$ for (9), we immediately see that this ellipse $\Phi_2(f_X)$ must be a two-dimensional ℓ_2-sphere. Furthermore, using the fact that f_X is a symmetric log-concave density, we can write it in the form

$$f_X(x) = c_1\, e^{-c_0 x^2}, \quad \text{where } c_1, c_0 > 0. \tag{19}$$

2) For $n > 2$, since (i) holds, let $\langle \cdot, \cdot \rangle_V$ be an inner product satisfying

$$\|\mathbf{x}\|_{\mathcal{K}} = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle_V}, \quad \forall \mathbf{x} \in V. \tag{20}$$

Let $V' = \operatorname{span}(\{\mathbf{e}_1, \mathbf{e}_2\})$, then

$$\langle \mathbf{x}', \mathbf{y}' \rangle_{V'} \triangleq \langle \mathbf{x}', \mathbf{y}' \rangle_V, \quad \forall \mathbf{x}', \mathbf{y}' \in V', \tag{21}$$

is an inner product on this subspace. Thus, $(V', \langle \cdot, \cdot \rangle_{V'})$ is an inner product space with the unit sphere

$$\mathcal{S}' = \Phi_n(f_X) \cap V' \tag{22}$$

$$= \left\{ \mathbf{x} \in \mathbb{R}^2 \colon \frac{-1}{n} \left(\sum_{i=1}^{2} \log f_X(x_i) + \sum_{i=3}^{n} \log f_X(0) \right) = \mathrm{h}(f_X) \right\}. \tag{23}$$

Since $\dim V' = 2$, it follows from Theorem 17 that $\mathcal{S}'$ is an ellipse, and therefore, similarly to 1), we can write f_X in the form of (19).

Combining 1) and 2) then yields (19) for any $n \geq 2$. Using (19), (10) and that X, Z are independent, we obtain that for any $t > 0$,

$$\Phi_n(f_{X+Z}) = \sqrt{1+t^2}\, \Phi_n(f_X). \tag{24}$$

Letting $\alpha_t = \sqrt{1+t^2} > 0$ in (24) we conclude (i) $\Longrightarrow$ (ii) for any $n \geq 2$.

Appendix B
Proof of Theorem 12

The proof consists of three parts that together prove the equivalence of Conditions (i)–(iii). The standard dot-product for $\mathbf{x}, \mathbf{y} \in V$ is denoted $\mathbf{x} \cdot \mathbf{y}$ and defined as

$$V \times V \to \mathbb{R},\ (\mathbf{x}, \mathbf{y}) \mapsto \sum_{i \in [n]} x_i\, y_i, \tag{25}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{y} = (y_1, y_2, \ldots, y_n)$.

A. (ii) $\Longrightarrow$ (i)

We start proving (ii) $\Longrightarrow$ (19). Pick an arbitrary $X \in \mathcal{F}$ and define $Y_0 = X$, let $k \in \mathbb{N}$, and for all $i \in [k]$, let Y_{i-1}, Z_i be independent RVs, where $\{Z_i\}$ are IID and where

$$Y_i = Y_{i-1} + Z_i. \tag{26}$$

By Proposition 9, Y_i has log-concave (symmetric) density, so we can apply (10) and (ii) recursively (starting from $i = 1$) to obtain a sequence of $\{\alpha_{t_i}\}_{i\in[k]}$ satisfying

$$\Phi_n(f_{Y_i}) = \alpha_{t_i}\Phi_n(f_{Y_{i-1}}) \quad \forall i \in [k]. \tag{27}$$

From this we then obtain

$$\Phi_n(f_{Y_k}) = \left(\prod_{i=1}^{k} \alpha_{t_i}\right)\Phi_n(f_{Y_0}). \tag{28}$$

By letting $\beta_k \triangleq 1/\prod_{i=1}^{k} \alpha_{t_i}$, we can rewrite (28) as

$$\Phi_n(f_X) = \beta_k\Phi_n(f_{Y_k}), \quad \beta_k > 0. \tag{29}$$

Note that by the central limit theorem, $Y_k/k = X/k + \frac{1}{k}\sum_{i=1}^{k} Z_i$ converges to a zero-mean Gaussian RV as k becomes sufficiently large. (Note that the central limit theorem applies because a log-concave density has finite moments of all orders and thus also finite variance.) Also, for a Gaussian RV U it holds that $\Phi_n(f_{U/k}) = \gamma_k\Phi_n(f_U)$ for some factor γ_k. Thus, and since (29) holds for all $k \in \mathbb{N}$, we take $k \to \infty$ and conclude that $\Phi_n(f_X)$ is an ℓ_2-sphere and that f_X either takes the form of (19) or is a Dirac delta. However the latter is excluded due to the continuity assumption of f_X.

Next, we are going to prove that (19) $\implies$ (i). Using (19) and the definition of (9) we obtain

$$\Phi_n(f_X) = \left\{\mathbf{x} \in \mathbb{R}^n \colon \sum_{i=1}^{n} x_i^2 = \frac{n}{2c_0}\right\} \tag{30}$$

to be the unit sphere of the normed space $(V, \|\cdot\|_{\mathcal{K}})$. Now we define an inner product $\langle\cdot,\cdot\rangle$ on V (which satisfies all five properties in Definition 4) as

$$\langle \mathbf{x}, \mathbf{y}\rangle \triangleq \frac{\mathbf{x}\cdot\mathbf{y}}{r_0^2}, \quad \forall \mathbf{x}, \mathbf{y} \in V, \tag{31}$$

with $r_0 \triangleq \sqrt{\frac{n}{2c_0}}$. Then by applying (1),

$$\|\mathbf{x}\|_{\mathcal{K}} = \inf\{\lambda > 0 \colon \mathbf{x} \in \lambda\mathcal{B}_n(f_X)\}$$
$$\implies \|\mathbf{x}\|_{\mathcal{K}}\mathbf{v} = \mathbf{x} \quad \text{where } \mathbf{v} \in \Phi_n(f_X) \tag{32}$$
$$\implies \frac{\mathbf{x}}{\|\mathbf{x}\|_{\mathcal{K}}} \in \Phi_n(f_X). \tag{33}$$

Using (33), (30) and (31) we get

$$\|\mathbf{x}\|_{\mathcal{K}} = \sqrt{\frac{\sum_{i=1}^{n} {x_i}^2}{r_0^2}} = \sqrt{\frac{\mathbf{x}\cdot\mathbf{x}}{r_0^2}} = \sqrt{\langle\mathbf{x},\mathbf{x}\rangle}, \tag{34}$$

i.e., $\|\mathbf{x}\|_{\mathcal{K}} = \sqrt{\langle\mathbf{x},\mathbf{x}\rangle}$, $\forall \mathbf{x} \in V$. Therefore, by Definition 6, we say that "$(V, \|\cdot\|_{\mathcal{K}})$ is an inner product space", which holds for arbitrary $X \in \mathcal{F}$ and $\mathcal{K} = \Phi_n(f_X)$, concluding our proof.

B. (i) $\implies$ (iii) (by way of (ii))

Since (i) holds, there exists an inner product $\langle\cdot,\cdot\rangle_{\mathcal{K}}$ such that $\|\mathbf{x}\|_{\mathcal{K}} = \sqrt{\langle\mathbf{x},\mathbf{x}\rangle_{\mathcal{K}}}$, $\forall \mathbf{x} \in V$. Using this and the properties of the inner product we can write

$$\left\|\frac{1}{\alpha_t}\mathbf{y} - \alpha_t\mathbf{x}\right\|_{\mathcal{K}}^2 = \left\langle \frac{1}{\alpha_t}\mathbf{y} - \alpha_t\mathbf{x}, \frac{1}{\alpha_t}\mathbf{y} - \alpha_t\mathbf{x}\right\rangle_{\mathcal{K}} \tag{35}$$
$$= \alpha_t^2\langle\mathbf{x},\mathbf{x}\rangle_{\mathcal{K}}^2 + \frac{\langle\mathbf{y},\mathbf{y}\rangle_{\mathcal{K}}^2}{\alpha_t^2} - 2\langle\mathbf{x},\mathbf{y}\rangle_{\mathcal{K}} \tag{36}$$
$$= \alpha_t^2\|\mathbf{x}\|_{\mathcal{K}}^2 + \frac{\|\mathbf{y}\|_{\mathcal{K}}^2}{\alpha_t^2} - 2\langle\mathbf{x},\mathbf{y}\rangle_{\mathcal{K}} \tag{37}$$
$$= \alpha_t^2 + \frac{\|\mathbf{y}\|_{\mathcal{K}}^2}{\alpha_t^2} - 2\langle\mathbf{x},\mathbf{y}\rangle_{\mathcal{K}}. \tag{38}$$

Similarly we have

$$\|\mathbf{y}-\mathbf{x}\|_{\mathcal{K}}^2 = \langle\mathbf{y}-\mathbf{x},\mathbf{y}-\mathbf{x}\rangle_{\mathcal{K}} = 1 + \|\mathbf{y}\|_{\mathcal{K}}^2 - 2\langle\mathbf{x},\mathbf{y}\rangle_{\mathcal{K}}. \tag{39}$$

Because in Section A-B we already proved (i) $\implies$ (ii), we see that (ii) holds, which implies $\|\mathbf{y}\|_{\mathcal{K}}^2 = \alpha_t^2$, where $\alpha_t > 0$ only depends on t. Applying this to (38) and (39) we get

$$\left\|\frac{1}{\alpha_t}\mathbf{y} - \alpha_t\mathbf{x}\right\|_{\mathcal{K}}^2 = \alpha_t^2 + 1 - 2\langle\mathbf{x},\mathbf{y}\rangle_{\mathcal{K}} = \|\mathbf{y}-\mathbf{x}\|_{\mathcal{K}}^2, \tag{40}$$

where $\alpha_t > 0$ only depends on t. Using Property (1) of the norm in Definition 3 for (40) we obtain (12), which concludes the proof.

C. (iii) $\implies$ (ii)

Since (12) holds for any $\mathbf{x} \in \Phi_n(f_X)$, $\mathbf{y} \in \Phi_n(f_{X+Z})$, we know that (12) also holds for any $\mathbf{y}_0 \in \Phi_n(f_{X+Z})$ and $\mathbf{x}_0 \triangleq \mathbf{y_0}/\|\mathbf{y}_0\|_{\mathcal{K}} \in \Phi_n(f_X)$. Taking $\mathbf{x} = \mathbf{x}_0$, $\mathbf{y} = \mathbf{y}_0$ in (12) we obtain

$$\left\|\mathbf{y}_0 - \frac{\mathbf{y}_0}{\|\mathbf{y}_0\|_{\mathcal{K}}}\right\|_{\mathcal{K}} = \left\|\frac{1}{\alpha_t}\mathbf{y}_0 - \frac{\alpha_t}{\|\mathbf{y}_0\|_{\mathcal{K}}}\mathbf{y}_0\right\|_{\mathcal{K}}$$
$$\implies \left|1 - \frac{1}{\|\mathbf{y}_0\|_{\mathcal{K}}}\right| = \left|\frac{1}{\alpha_t} - \frac{\alpha_t}{\|\mathbf{y}_0\|_{\mathcal{K}}}\right| \tag{41}$$
$$\implies \|\mathbf{y}_0\|_{\mathcal{K}} = \alpha_t, \quad \text{where } \alpha_t > 0. \tag{42}$$

In (42) $\|\mathbf{y}_0\|_{\mathcal{K}} = -\alpha_t$ is not valid because $\|\mathbf{y}_0\|_{\mathcal{K}} \geq 0$.

Because (42) holds for any $\mathbf{y}_0 \in \Phi_n(f_{X+Z})$ and $\Phi_n(f_X)$ is the unit sphere of $(V, \|\cdot\|_{\mathcal{K}})$, we obtain

$$\Phi_n(f_{X+Z}) = \alpha_t\Phi_n(f_X), \tag{43}$$

which concludes the proof.

Acknowledgment

We thank Dr. Simone Carlo Surace for drawing the authors' attention to the optimal Gaussian source channel communication system, which helped commence this project.

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