

# The Fading Number of Memoryless Multiple-Input Multiple-Output Fading Channels

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## 1 Channel Model

We consider a multiple-input multiple-output fading channel:

$$\mathbf{Y} = \mathbb{H}\mathbf{x} + \mathbf{Z}$$

where

- $\mathbf{Y} \in \mathbb{C}^{n_R}$  denotes the random channel output vector ( $n_R$  **receive antennas**);
- $\mathbf{Z} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \sigma^2 \mathbf{I}_{n_R})$  denotes additive white (complex) **Gaussian noise**;
- $\mathbf{x} \in \mathbb{C}^{n_T}$  denotes the channel input vector ( $n_T$  **transmit antennas**) satisfying either a peak-power or an average-power constraint  $\mathcal{E}$  with signal-to-noise ratio  $\text{SNR} \triangleq \frac{\mathcal{E}}{\sigma^2}$ ;
- $\mathbb{H}$  denotes the random  $n_R \times n_T$  fading matrix of **general** (not necessarily Gaussian!) **law**: we only assume that  $\mathbb{H}$  has finite second moment  $\mathbb{E}[\|\mathbb{H}\|_{\text{F}}^2] < \infty$  and finite differential entropy  $h(\mathbb{H}) > -\infty$  (the **regularity** assumption). Note that the components of  $\mathbb{H}$  may be dependent (**spatial memory**). The realizations of  $\mathbb{H}$  are **unknown** both at transmitter and receiver (**non-coherent decoding**).

## 2 Channel Capacity

We know that for such regular, non-coherent fading channels the capacity grows double-logarithmically at very high power ( $\text{SNR} \rightarrow \infty$ ):

$$C(\text{SNR}) = \log(1 + \log(1 + \text{SNR})) + \chi(\mathbb{H}) + o(1)$$

where  $o(1)$  tends to zero as  $\text{SNR} \rightarrow \infty$  and where  $\chi(\mathbb{H})$  is a constant called **fading number** that is independent of SNR, but depends on the distribution of  $\mathbb{H}$ .

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## 3 Main Results

**Lemma 1.** *Any capacity-achieving input distribution must **escape to infinity**. This means that for  $\text{SNR} \rightarrow \infty$  the capacity-achieving input distributions will not use any finite-cost input symbols.*

**Lemma 2.** *The capacity-achieving input can be assumed to be **circularly symmetric**, i.e., the input  $\mathbf{X}$  can be replaced by  $\mathbf{X}e^{i\Theta}$ , where  $\Theta \sim \mathcal{U}([0, 2\pi])$  and is independent of every other random quantity.*

**Theorem 3.** *The memoryless multiple-input multiple-output (MIMO) fading number  $\chi(\mathbb{H})$  is given by*

$$\chi(\mathbb{H}) = \sup_{Q_{\hat{\mathbf{X}}}} \left\{ h_{\lambda} \left( \frac{\mathbb{H}\hat{\mathbf{X}}}{\|\mathbb{H}\hat{\mathbf{X}}\|} \right) + n_R \mathbb{E} \left[ \log \|\mathbb{H}\hat{\mathbf{X}}\|^2 \right] - \log 2 - h(\mathbb{H}\hat{\mathbf{X}} | \hat{\mathbf{X}}) \right\}$$

where the supremum is taken over all distributions of the **random unit vector**  $\hat{\mathbf{X}}$ . Note that  $h_{\lambda}(\cdot)$  denotes a differential entropy for random vectors that take value on the unit sphere in  $\mathbb{C}^{n_R}$ .

Moreover, this fading number is **achievable by** a random vector  $\mathbf{X} = \hat{\mathbf{X}} \cdot R$  where  $\hat{\mathbf{X}}$  is distributed according to the distribution that achieves the fading number above and where  $R$  is a non-negative random variable independent of  $\hat{\mathbf{X}}$  such that

$$\log R^2 \sim \mathcal{U}([\log \log \mathcal{E}, \log \mathcal{E}]).$$

## References

- [1] S. M. Moser, "The fading number of memoryless multiple-input multiple-output fading channels," *IEEE Transactions on Information Theory*, vol. 53, no. 7, pp. 2652–2666, July 2007.