Some Expectations of a Non-Central Chi-Square Distribution With an Even Number of Degrees of Freedom

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April 27, 2007

Abstract

The non-central chi-square distribution plays an important role in communications, for example in the analysis of mobile and wireless communication systems. It not only includes the important cases of a squared Rayleigh distribution and a squared Rice distribution, but also the generalizations to a sum of independent squared Gaussian random variables of identical variance with or without mean, *i.e.*, a "squared MIMO Rayleigh" and "squared MIMO Rice" distribution.

In this paper closed-form expressions are derived for the expectation of the logarithm and for the expectation of the n-th power of the reciprocal value of a non-central chi-square random variable. It is shown that these expectations can be expressed by a family of continuous functions $g_m(\cdot)$ and that these families have nice properties (monotonicity, convexity, etc.). Moreover, some tight upper and lower bounds are derived that are helpful in situations where the closed-form expression of $g_m(\cdot)$ is too complex for further analysis.

Index Terms: Non-central chi-square distribution, Rayleigh, Rice, expected logarithm, expected reciprocal value.

1 Introduction

It is well known that adding several independent squared Gaussian random variables of identical variance yields a random variable that is non-central chi-square distributed. This distribution often shows up in information theory and communications. As an example we mention the situation of a non-coherent multiple-input multiple-output (MIMO) fading channel

$$\mathbf{Y} = \mathbb{H}\mathbf{x} + \mathbf{Z} \tag{1}$$

with an additive white Gaussian noise vector $\mathbf{Z}$ and with a fading matrix $\mathbb{H}$ that consists of independent and unit-variance Gaussian distributed components with or without mean. Here conditional on the input $\mathbf{x}$ the squared magnitude of the output $\|\mathbf{Y}\|^2$ is non-central chi-square distributed.

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While the special case of a squared Rayleigh distribution[1] is well understood in the sense that there exist closed-form expressions for more or less all interesting expected values, the more general situation of a non-central chi-square distribution is far more complex. Here, standard integration tools (*e.g.*, *Maple*) or integration lookup tables (*e.g.*, [2]) will very quickly cease to provide closed-form expressions.

In this paper we will state closed-form expressions for some of these situations: we will give closed-form solutions to $\mathsf{E}[\ln V]$ and $\mathsf{E}\left[\frac{1}{V^n}\right]$ for a non-central chi-square random variable V with an even number of degrees of freedom. Note that in practice we often have an even number of degrees of freedom because we usually consider *complex* Gaussian random variables consisting of *two real* Gaussian components. We will see that these expectations are all related to a family of functions $g_m(\cdot)$ that is defined in Definition 2 in the following section. There we will also state the main results. In Section 3 we will then derive some properties of the functions $g_m(\cdot)$ and in Section 4 we will state tight upper and lower bounds. In Section 5 another property of $g_m(\cdot)$ is derived as an example of how the bounds from Section 4 could be applied. We conclude in Section 6.

2 Definitions and Main Results

A non-negative real random variable is said to have a *non-central chi-square* distribution with n *degrees of freedom* and *non-centrality parameter* s^2 if it is distributed like

$$\sum_{j=1}^{n} (X_j + \mu_j)^2, \tag{2}$$

where $\{X_j\}_{j=1}^n$ are IID $\sim \mathcal{N}_{\mathbb{R}}(0,1)$ and the real constants $\{\mu_j\}_{j=1}^n$ satisfy

$$s^2 = \sum_{j=1}^{n} \mu_j^2. \tag{3}$$

(The distribution of (2) depends on the constants $\{\mu_j\}$ only via the sum of their squares.) The probability density function of such a distribution is given by [3, Chapter 29]

$$\frac{1}{2}\left(\frac{x}{s^2}\right)^{\frac{n-2}{4}} e^{-\frac{s^2+x}{2}} I_{n/2-1}\left(s\sqrt{x}\right), \qquad x \geq 0. \tag{4}$$

Here $I_\nu(\cdot)$ denotes the modified Bessel function of the first kind of order $\nu \in \mathbb{R}$, *i.e.*,

$$I_\nu(x) \triangleq \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{\nu+2k}, \qquad x \geq 0 \tag{5}$$

(see [2, Equation 8.445]).

If the number of degrees of freedom n is even, *i.e.*, if $n = 2m$ for some positive integer m, then the non-central chi-square distribution can also be expressed as a sum of the squared norms of *complex* Gaussian random variables:

Definition 1. *Let the random variable V have a non-central chi-square distribution with an even number $2m$ of degrees of freedom,* i.e.*,*

$$V \triangleq \sum_{j=1}^{m} \left|U_j + \mu_j\right|^2 \tag{6}$$

[1]A squared Rayleigh random variable is actually exponentially distributed.

where $\{U_j\}_{j=1}^m$ *are IID* $\sim \mathcal{N}_{\mathbb{C}}(0,1)$*, and* $\{\mu_j\}_{j=1}^m$ *are complex constants. Let further the non-centrality parameter* s^2 *be defined as*

$$s^2 \triangleq \sum_{j=1}^{m} |\mu_j|^2. \tag{7}$$

Next we define the following continuous functions:

Definition 2. *The functions* $g_m(\cdot)$ *are defined as follows:*

$$g_m(\xi) \triangleq \begin{cases} \ln(\xi) - \mathrm{Ei}(-\xi) + \displaystyle\sum_{j=1}^{m-1} (-1)^j \Bigg[e^{-\xi}(j-1)! \\ \qquad - \dfrac{(m-1)!}{j(m-1-j)!} \Bigg] \left(\dfrac{1}{\xi}\right)^j, & \xi > 0 \\ \psi(m), & \xi = 0 \end{cases} \tag{8}$$

for $m \in \mathbb{N}$*, where* $\mathrm{Ei}(\cdot)$ *denotes the exponential integral function defined as*

$$\mathrm{Ei}(-x) \triangleq -\int_x^\infty \frac{e^{-t}}{t}\,\mathrm{d}t, \qquad x > 0 \tag{9}$$

and $\psi(\cdot)$ *is Euler's psi function given by*

$$\psi(m) \triangleq -\gamma + \sum_{j=1}^{m-1} \frac{1}{j} \tag{10}$$

with $\gamma \approx 0.577$ *denoting Euler's constant.*

Note that $g_m(\xi)$ is continuous for all $\xi \geq 0$, *i.e.*, in particular

$$\lim_{\xi \downarrow 0} \left\{ \ln(\xi) - \mathrm{Ei}(-\xi) + \sum_{j=1}^{m-1} (-1)^j \left[e^{-\xi}(j-1)! - \frac{(m-1)!}{j(m-1-j)!} \right] \left(\frac{1}{\xi}\right)^j \right\} = \psi(m) \tag{11}$$

for all $m \in \mathbb{N}$. Therefore its first derivative is defined for all $\xi \geq 0$ and can be evaluated to

$$g'_m(\xi) \triangleq \frac{\partial g_m(\xi)}{\partial \xi} = \frac{(-1)^m \Gamma(m)}{\xi^m} \left(e^{-\xi} - \sum_{j=0}^{m-1} \frac{(-1)^j}{j!} \xi^j \right) \tag{12}$$

(see [4, Eq. (417)], [5, Eq. (A.39)]). Note that $g'_m(\cdot)$ is also continuous, *i.e.*, in particular

$$\lim_{\xi \downarrow 0} \left\{ \frac{(-1)^m \Gamma(m)}{\xi^m} \left(e^{-\xi} - \sum_{j=0}^{m-1} \frac{(-1)^j}{j!} \xi^j \right) \right\} = \frac{1}{m} = g'_m(0). \tag{13}$$

Now we give a closed-form expression for a first expectation of a non-central chi-square random variable:

Theorem 3. *The expected value of the logarithm of a non-central chi-square random variable with an even number* $2m$ *of degrees of freedom is given as*

$$\boxed{\mathsf{E}[\ln V] = g_m(s^2),} \tag{14}$$

where V *and* s^2 *are defined in* (6) *and* (7)*. Hence, we have the solution to the following integral:*

$$\int_0^\infty \ln v \cdot \left(\frac{v}{s^2}\right)^{\frac{m-1}{2}} e^{-v-s^2} I_{m-1}(2s\sqrt{v})\,\mathrm{d}v = g_m(s^2) \tag{15}$$

for any $m \in \mathbb{N}$ *and* $s^2 \geq 0$.

Proof. A proof can be found in [4, Lem. 10.1], [5, Lem. A.6] □

Next we derive the following expectations:

Theorem 4. *Let* $n \in \mathbb{N}$ *with* $n < m$*. The expected value of the* n*-th power reciprocal value of a non-central chi-square random variable with an even number* $2m$ *of degrees of freedom is given as*

$$\boxed{\mathsf{E}\left[\frac{1}{V^n}\right] = \frac{(-1)^{n-1}}{(n-1!)} \cdot g_{m-n}^{(n)}(s^2), \qquad m > n} \tag{16}$$

where

$$g_m^{(\ell)}(\xi) = \frac{\partial^\ell g_m(\xi)}{\partial \xi^\ell} \tag{17}$$

denotes the ℓ*-th derivative of* $g_m(\cdot)$ *and where* V *and* s^2 *are defined in* (6) *and* (7)*. In particular, for* $m > 1$

$$\boxed{\mathsf{E}\left[\frac{1}{V}\right] = g'_{m-1}(s^2).} \tag{18}$$

Hence, we have the solution to the following integral:

$$\int_0^\infty \frac{1}{v^n} \cdot \left(\frac{v}{s^2}\right)^{\frac{m-1}{2}} e^{-v-s^2} I_{m-1}(2s\sqrt{v})\,\mathrm{d}v = \frac{(-1)^{n-1}}{(n-1!)} \cdot g_{m-n}^{(n)}(s^2) \tag{19}$$

for any $m, n \in \mathbb{N}$*,* $m > n$*, and any real* $s^2 \geq 0$.

Note that in the cases where $m \leq n$, the expectation is unbounded.

Proof. A proof can be found in Appendix A. □

3 Properties of $g_m(\cdot)$ and $g'_m(\cdot)$

In this section we will show that the family of functions $g_m(\cdot)$ and $g'_m(\cdot)$ are well-behaved.

Corollary 5. *The functions* $g_m(\cdot)$ *are monotonically strictly increasing and strictly concave in the interval* $[0, \infty)$ *for all* $m \in \mathbb{N}$.

Proof. From [4, Eqs. (415), (416)], [5, Eqs. (A.37), (A.38)] we know that

$$g'_m(\xi) = \frac{\partial g_m(\xi)}{\partial \xi} = e^{-\xi} \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{1}{k+m} \cdot \xi^k, \tag{20}$$

$$g''_m(\xi) = \frac{\partial^2 g_m(\xi)}{\partial \xi^2} = -e^{-\xi} \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{1}{(k+m)(k+m+1)} \cdot \xi^k, \tag{21}$$

i.e., the first derivative of $g_m(\cdot)$ is positive and the second derivative is negative. □

Corollary 6. *The function $g_m(\xi)$ is monotonically strictly increasing in m for all $\xi \geq 0$.*

Proof. Fix two arbitrary natural numbers $m_1, m_2 \in \mathbb{N}$ such that $m_1 < m_2$. Choose $\mu_1 = s$, $\mu_2 = \ldots = \mu_{m_2} = 0$, with an arbitrary $s \geq 0$. Let $\{U_j\}_{j=1}^{m_2}$ be IID $\sim \mathcal{N}_{\mathbb{C}}(0,1)$. Then

$$g_{m_2}(s^2) = \mathsf{E}\left[\ln\left(\sum_{j=1}^{m_2} |U_j + \mu_j|^2\right)\right] \tag{22}$$

$$= \mathsf{E}\left[\ln\left(\sum_{j=1}^{m_1} |U_j + \mu_j|^2 + \sum_{j=m_1+1}^{m_2} |U_j + \mu_j|^2\right)\right] \tag{23}$$

$$> \mathsf{E}\left[\ln\left(\sum_{j=1}^{m_1} |U_j + \mu_j|^2\right)\right] \tag{24}$$

$$= g_{m_1}(s^2), \tag{25}$$

where the first equality follows from (14); the subsequent equality from splitting the sum into two parts; the subsequent inequality from dropping some positive terms; and the final equality again from (14). □

Corollary 7. *The functions $g'_m(\cdot)$ are positive, monotonically strictly decreasing, and strictly convex functions for all $m \in \mathbb{N}$.*

Proof. The positivity and the monotonicity follow from (20) and (21). To see the convexity, compute from (21)

$$g'''_m(\xi) = \frac{\partial^3 g_m(\xi)}{\partial \xi^3} = e^{-\xi} \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{2}{(k+m)(k+m+1)(k+m+2)} \cdot \xi^k \tag{26}$$

which is positive. □

Corollary 8. *The function $g'_m(\xi)$ is monotonically strictly decreasing in m for all $\xi \geq 0$.*

Proof. Fix two arbitrary natural numbers $m_1, m_2 \in \mathbb{N}$ such that $m_2 > m_1 > 1$. Choose $\mu_1 = s$, $\mu_2 = \ldots = \mu_{m_2} = 0$, with an arbitrary $s \geq 0$. Let $\{U_j\}_{j=1}^{m_2}$ be IID

$\sim \mathcal{N}_{\mathbb{C}}(0,1)$. Then

$$g'_{m_2-1}(s^2) = \mathsf{E}\left[\frac{1}{\sum_{j=1}^{m_2} |U_j + \mu_j|^2}\right] \tag{27}$$

$$= \mathsf{E}\left[\frac{1}{\sum_{j=1}^{m_1} |U_j + \mu_j|^2 + \sum_{j=m_1+1}^{m_2} |U_j + \mu_j|^2}\right] \tag{28}$$

$$< \mathsf{E}\left[\frac{1}{\sum_{j=1}^{m_1} |U_j + \mu_j|^2}\right] \tag{29}$$

$$= g'_{m_1-1}(s^2), \tag{30}$$

where the first equality follows from (18); the subsequent equality from splitting the sum into two parts; the subsequent inequality from dropping some positive terms in the denominator; and the final equality again from (18). □

Theorem 9. *We have the following relation:*

$$g_{m+1}(\xi) = g_m(\xi) + g'_m(\xi) \tag{31}$$

for all $m \in \mathbb{N}$ *and all* $\xi \geq 0$.

Proof. A proof is given in Appendix B. □

Theorem 10. *We have the following relation:*

$$g'_{m+1}(\xi) = \frac{1}{\xi} - \frac{m}{\xi} g'_m(\xi) \tag{32}$$

for all $m \in \mathbb{N}$ *and all* $\xi \geq 0$.

Proof. A proof is given in Appendix C. □

4 Bounds on $g_m(\cdot)$ and $g'_m(\cdot)$

In this section we derive some tight bounds on the functions $g_m(\cdot)$ and $g'_m(\cdot)$.

Theorem 11. *The function* $g'_m(\cdot)$ *can be bounded as follows:*

$$\frac{1}{\xi + m} \leq g'_m(\xi) \leq \min\left\{\frac{m+1}{m(\xi + m + 1)}, \frac{1}{\xi + m - 1}\right\}. \tag{33}$$

Note that for $\xi < m+1$ *the first of the two upper bounds is tighter than second, while for* $\xi > m+1$ *the second is tighter. Moreover, the first upper bound coincides with* $g_m(\xi)$ *for* $\xi = 0$*, and the second upper bound is asymptotically tight when* ξ *tends to infinity.*

Proof. A proof is given in Appendix D. □

The bounds (33) are depicted in Figure 1 for the cases of $m = 1$, $m = 3$, and $m = 8$.

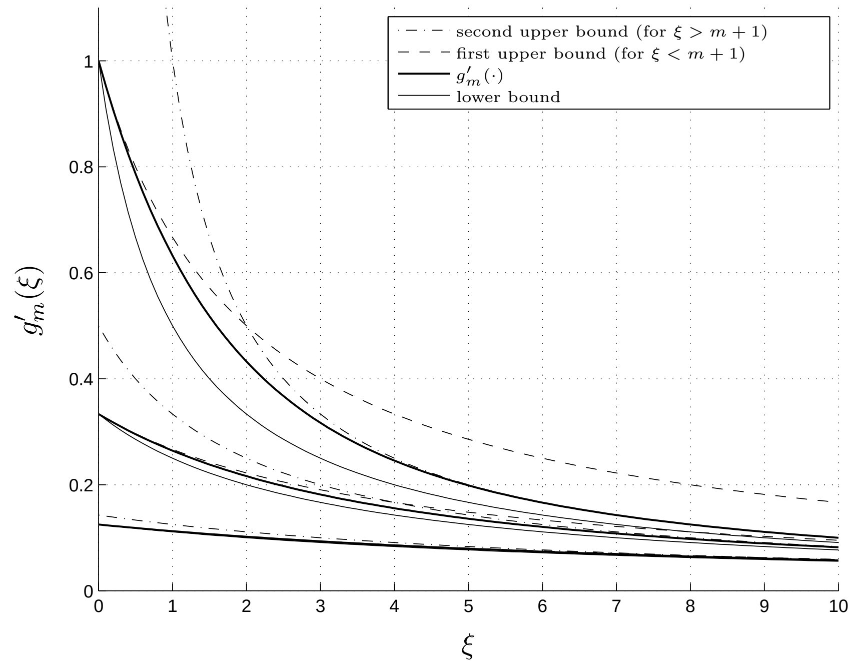

Figure 1: Upper and lower bounds on $g'_m(\cdot)$ according to Theorem 11. The top four curves correspond to $m = 1$, the middle four to $m = 3$, and the lowest group of four curves to $m = 8$.

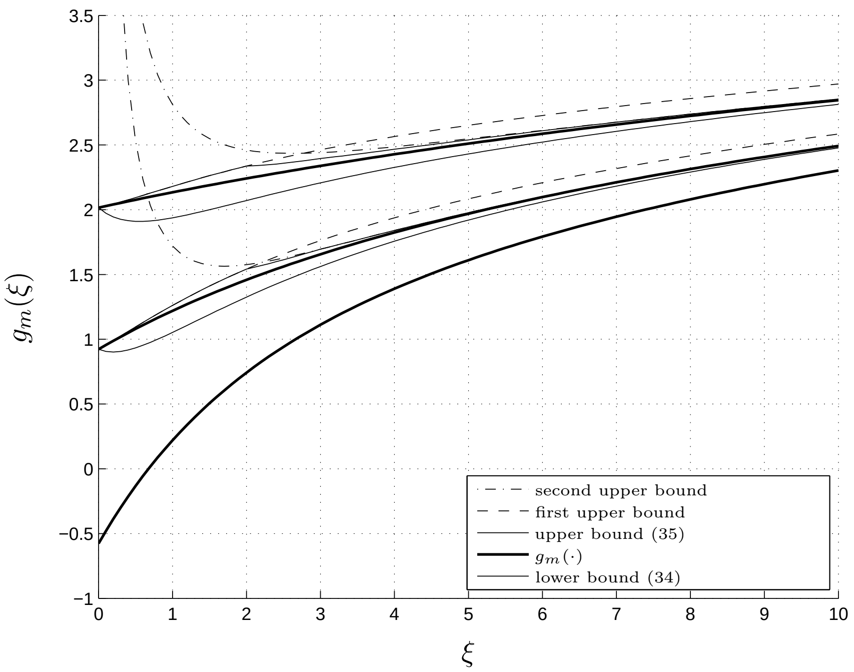

Figure 2: Upper and lower bounds on $g_m(\cdot)$ according to (34) and (35) in Theorem 12. The lowest curve corresponds to $m = 1$ (in this case all bounds coincides with $g_1(\cdot)$), the next five curves correspond to $m = 3$, and the top five to $m = 8$.

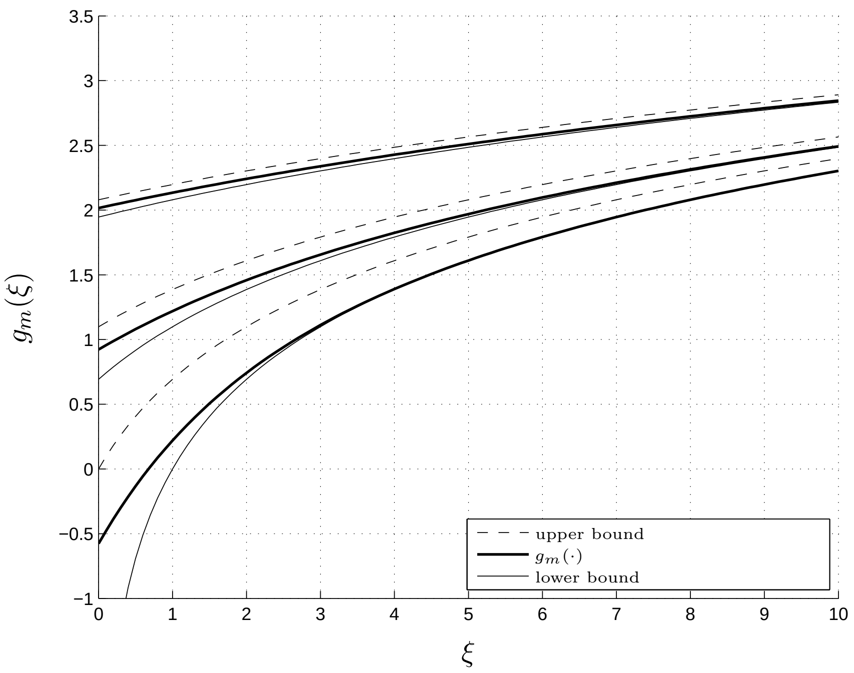

Figure 3: Upper and lower bounds on $g_m(\cdot)$ according to (36) in Theorem 12. The lowest three curves correspond to $m = 1$, the next three to $m = 3$, and the top three to $m = 8$.

Theorem 12. *For the functions $g_m(\cdot)$ we state two sets of bounds. The first set is tighter for smaller values of m:*

$$g_m(\xi) \geq \ln \xi - \operatorname{Ei}(-\xi) + \sum_{j=1}^{m-1} \frac{1}{\xi + j}, \tag{34}$$

$$g_m(\xi) \leq \ln \xi - \operatorname{Ei}(-\xi) + \sum_{j=1}^{m-1} \min\left\{ \frac{j+1}{j(\xi + j + 1)}, \frac{1}{\xi + j - 1} \right\}. \tag{35}$$

Secondly, we give a set of bounds that is tight for large values of m:

$$\ln(\xi + m - 1) \leq g_m(\xi) \leq \ln(\xi + m). \tag{36}$$

Note that this second set of bounds is very simple, i.e., *it is particularly interesting for further analysis.*

Proof. A proof is given in Appendix E. □

The bounds (34) and (35) are depicted in Figure 2 and the bounds (36) in Figure 3, both times for the cases of $m = 1$, $m = 3$, and $m = 8$.

5 Additional Properties

As an example of how the properties given in Section 3 and the bounds given in Section 4 can be used to derive further results we state the following corollary:

Corollary 13. *The functions* $g_m\left(\frac{1}{\xi}\right)$ *are monotonically strictly decreasing and convex in* ξ *for all* $m \in \mathbb{N}$.

Proof. We start as follows:

$$\frac{\partial}{\partial \xi} g_m\left(\frac{1}{\xi}\right) = -\frac{1}{\xi^2} \cdot g'_m\left(\frac{1}{\xi}\right) \tag{37}$$

$$= -\frac{1}{\xi^2}\left(g_{m+1}\left(\frac{1}{\xi}\right) - g_m\left(\frac{1}{\xi}\right)\right), \tag{38}$$

where the last equality follows from Theorem 9. From Corollary 7 we know that $g'_m(\cdot) > 0$, so that we can conclude from (37) that $g_m\left(\frac{1}{\xi}\right)$ is monotonically strictly decreasing.

To check convexity, we continue as follows:

$$\frac{\partial^2}{\partial \xi^2} g_m\left(\frac{1}{\xi}\right) = \frac{2}{\xi^3} g_{m+1}\left(\frac{1}{\xi}\right) + \frac{1}{\xi^4} g'_{m+1}\left(\frac{1}{\xi}\right) - \frac{2}{\xi^3} g_m\left(\frac{1}{\xi}\right) - \frac{1}{\xi^4} g'_m\left(\frac{1}{\xi}\right) \tag{39}$$

$$= \frac{2}{\xi^3} g'_m\left(\frac{1}{\xi}\right) + \frac{\xi - m\xi g'_m\left(\frac{1}{\xi}\right) - g'_m\left(\frac{1}{\xi}\right)}{\xi^4} \tag{40}$$

$$= g'_m\left(\frac{1}{\xi}\right) \cdot \frac{2\xi - m\xi - 1}{\xi^4} + \frac{1}{\xi^3} \tag{41}$$

$$\geq \frac{1}{\frac{1}{\xi} + m} \cdot \frac{2\xi - m\xi - 1}{\xi^4} + \frac{1}{\xi^3} \tag{42}$$

$$= \frac{2}{\xi^2(m\xi + 1)}. \tag{43}$$

Here, in the second equality we use Theorems 9 and 10; and the inequality follows from the lower bound of Theorem 11. Hence, the second derivative is positive and the statement proved. □

6 Conclusions

We have derived closed-form expressions for some important expectations in the field of information theory and communications. We have shown that the resulting functions behave nicely and we have given tight upper and lower bounds to them.

For brevity we will not include a detailed example of how these results can be used, but only mention that the derivation of the fading number of a non-coherent MIMO Gaussian fading channel [6] strongly depends on them. There are many more examples.

A Proof of Theorem 4

Let $n \in \mathbb{N}$ be arbitrary and assume that $m > n$. Then the required expectation can be written as

$$\mathsf{E}\left[\frac{1}{V^n}\right] = \int_0^\infty \frac{1}{v^n} \cdot \left(\frac{v}{s^2}\right)^{\frac{m-1}{2}} e^{-v-s^2} I_{m-1}(2s\sqrt{v})\, \mathrm{d}v. \tag{44}$$

Expressing $I_{m-1}(\cdot)$ as a power series (5)

$$I_{m-1}(z) = \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(m+k)} \left(\frac{z}{2}\right)^{m-1+2k} \tag{45}$$

we obtain from [2, Eq. 3.351-3] (using that $m > n$)

$$\mathsf{E}\left[\frac{1}{V^n}\right] = \frac{1}{s^{m-1}} e^{-s^2} \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(m+k)} s^{2k+m-1} \cdot \int_0^{\infty} v^{k+m-1-n} e^{-v}\,\mathrm{d}v \tag{46}$$

$$= e^{-s^2} \sum_{k=0}^{\infty} \frac{1}{k!\,(m+k-1)!} s^{2k} \cdot (k+m-1-n)! \tag{47}$$

$$= e^{-s^2} \sum_{k=0}^{\infty} \frac{1}{k!\,(k+m-1)\cdots(k+m-n)} s^{2k}. \tag{48}$$

Generalizing (20), (21), and (26) to the ℓ-th derivative, we have

$$g_m^{(\ell)}(\xi) = \frac{\partial^\ell g_m(\xi)}{\partial \xi^\ell} = (-1)^{\ell-1} e^{-\xi} \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{(\ell-1)!}{(k+m)\cdots(k+m+\ell-1)} \cdot \xi^k. \tag{49}$$

Comparing this with (48) we see that

$$\mathsf{E}\left[\frac{1}{V^n}\right] = \frac{(-1)^{n-1}}{(n-1!)} \cdot g_{m-n}^{(n)}(s^2). \tag{50}$$

B Proof of Theorem 9

Using (8) and (12) we get

$$\begin{aligned}
&g_m(\xi) + g_m'(\xi) \\
&\quad= \ln(\xi) - \mathrm{Ei}(-\xi) + \sum_{j=1}^{m-1} (-1)^j e^{-\xi} (j-1)! \left(\frac{1}{\xi}\right)^j \\
&\qquad - \sum_{j=1}^{m-1} (-1)^j \frac{(m-1)!}{j(m-1-j)!} \left(\frac{1}{\xi}\right)^j + \frac{(-1)^m (m-1)!}{\xi^m} e^{-\xi} \\
&\qquad - \sum_{i=0}^{m-1} (-1)^{i+m} \frac{(m-1)!}{i!} \xi^{i-m} &(51)\\
&\quad= \ln(\xi) - \mathrm{Ei}(-\xi) + \sum_{j=1}^{m} (-1)^j e^{-\xi} (j-1)! \left(\frac{1}{\xi}\right)^j \\
&\qquad - \sum_{j=1}^{m-1} (-1)^j \frac{(m-1)!}{j(m-1-j)!} \left(\frac{1}{\xi}\right)^j - \sum_{j=1}^{m} \underbrace{(-1)^{-j+2m}}_{=(-1)^j} \frac{(m-1)!}{(m-j)!} \xi^{-j} &(52)\\
&\quad= \ln(\xi) - \mathrm{Ei}(-\xi) + \sum_{j=1}^{m} (-1)^j e^{-\xi} (j-1)! \left(\frac{1}{\xi}\right)^j \\
&\qquad - \sum_{j=1}^{m-1} (-1)^j (m-1)! \underbrace{\left(\frac{1}{j(m-1-j)!} + \frac{1}{(m-j)!}\right)}_{=\frac{m}{j(m-j)!}} \left(\frac{1}{\xi}\right)^j \\
&\qquad - (-1)^m \frac{(m-1)!}{0!} \left(\frac{1}{\xi}\right)^m &(53)\\
&\quad= \ln(\xi) - \mathrm{Ei}(-\xi) + \sum_{j=1}^{m} (-1)^j e^{-\xi} (j-1)! \left(\frac{1}{\xi}\right)^j
\end{aligned}$$

$$- \sum_{j=1}^{m-1} (-1)^j (m-1)! \frac{m}{j(m-j)!} \left(\frac{1}{\xi}\right)^j - (-1)^m \frac{m!}{m\, 0!} \left(\frac{1}{\xi}\right)^m \tag{54}$$

$$= \ln(\xi) - \mathrm{Ei}(-\xi) + \sum_{j=1}^{m} (-1)^j e^{-\xi} (j-1)! \left(\frac{1}{\xi}\right)^j$$

$$- \sum_{j=1}^{m} (-1)^j \frac{m!}{j(m-j)!} \left(\frac{1}{\xi}\right)^j \tag{55}$$

$$= g_{m+1}(\xi). \tag{56}$$

Here, the first equality follows from the definitions given in (8) and (12); in the subsequent equality we combine the second last term with the first sum and reorder the last summation by introducing a new counter-variable $j = m - i$; the subsequent three equalities follows from arithmetic rearrangements; and the final equality follows again from definition (8).

C Proof of Theorem 10

Using (20) we get

$$\frac{1}{\xi} - \frac{m}{\xi} g'_m(\xi) = \frac{1}{\xi} e^{-\xi} \cdot e^{\xi} - \frac{m}{\xi} e^{-\xi} \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{1}{k+m} \cdot \xi^k \tag{57}$$

$$= \frac{1}{\xi} e^{-\xi} \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \xi^k - \frac{1}{\xi} e^{-\xi} \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{m}{k+m} \cdot \xi^k \tag{58}$$

$$= \frac{1}{\xi} e^{-\xi} \sum_{k=0}^{\infty} \frac{1}{k!} \left(1 - \frac{m}{k+m}\right) \xi^k \tag{59}$$

$$= e^{-\xi} \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{k}{k+m} \cdot \xi^{k-1} \tag{60}$$

$$= e^{-\xi} \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \cdot \frac{1}{k+m} \cdot \xi^{k-1} \tag{61}$$

$$= e^{-\xi} \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{1}{k+m+1} \cdot \xi^k \tag{62}$$

$$= g'_{m+1}(\xi). \tag{63}$$

Here, the first equality follows from (20); in the subsequent equality we use the series expansion of e^{ξ} which is valid for all $\xi \geq 0$; the subsequent two equalities follow from algebraic rearrangements; in the next equality we note that for $k = 0$ the terms in the sum are equal to zero; the second last equality then follows from renumbering the terms; and the last equality follows again from (20).

D Proof of Theorem 11

We start with the lower bound and note that for $\xi = 0$ the bound is tight:

$$g'_m(0) = g_{m+1}(0) - g_m(0) = \psi(m+1) - \psi(m) = \frac{1}{m} \tag{64}$$

where the first equality follows from Theorem 9, the second from (8) and the final from (10); and

$$\left.\frac{1}{\xi+m}\right|_{\xi=0} = \frac{1}{m}. \tag{65}$$

Moreover we notice that the bound is asymptotically tight, too:

$$\lim_{\xi\uparrow\infty} g'_m(\xi) = 0; \tag{66}$$

$$\lim_{\xi\uparrow\infty} \frac{1}{\xi+m} = 0. \tag{67}$$

Hence, since additionally both functions $g'_m(\cdot)$ and $\frac{1}{\cdot+m}$ are monotonically strictly decreasing and strictly convex, they cannot cross. Moreover, it is shown[2] in Theorem 12 that $g_m(\xi) \le \ln(\xi+m)$, so that we must have

$$g'_m(\xi) \ge \frac{\partial}{\partial\xi}\ln(\xi+m) = \frac{1}{\xi+m}. \tag{68}$$

Next we turn to the first upper bound which will follow from the lower bound (33) derived above together with Theorem 10:

$$g'_{m-1}(\xi) = \frac{1-\xi g'_m(\xi)}{m-1} \le \frac{1-\xi\frac{1}{\xi+m}}{m-1} = \frac{m}{(m-1)(\xi+m)}. \tag{69}$$

To derive the second upper bound we use once again Theorem 10 and the lower bound (33) derived above:

$$\frac{1}{\xi+m-1} - g'_m(\xi) = \frac{1}{\xi+m-1} - \frac{1}{\xi} + \frac{m-1}{\xi} g'_{m-1}(\xi) \tag{70}$$

$$\ge \frac{1}{\xi+m-1} - \frac{1}{\xi} + \frac{m-1}{\xi}\,\frac{1}{\xi+m-1} \tag{71}$$

$$= 0, \tag{72}$$

where the first equality follows from Theorem 10 and the inequality from the lower bound in (33).

E Proof of Theorem 12

The first set of bounds (34) and (35) follow directly from Theorems 9 and 11 and from the fact that

$$g_1(\xi) = \ln\xi - \mathrm{Ei}(-\xi)\,. \tag{73}$$

The upper bound in the second set of bounds (36) has been proven before in [7, App. B]. The proof is based on Jensen's inequality.

The lower bound in (36) follows from a slightly more complicated argument: Note that both $g_m(\cdot)$ and $\ln(\cdot+m-1)$ are monotonically strictly increasing and strictly concave (see Corollary 5). Hence, they can cross at most twice. Now asymptotically as $\xi\uparrow\infty$ the two functions coincide which corresponds to one of these "crossings." So they can only cross at most once more for finite ξ. For $\xi=0$, we have

$$g_m(0) = \psi(m) > \ln(m-1) \tag{74}$$

[2]Note that this part of the proof of Theorem 12 does not rely in any way on the theorem under consideration.

for all $m \in \mathbb{N}$ (where for $m = 1$ we take $\ln 0 = -\infty$). Let's assume for the moment that there is another crossing for a finite value ξ_0. Then, for $\xi > \xi_0$, $\ln(\xi + m - 1)$ is strictly larger than $g_m(\xi)$. However, since asymptotically they will coincide, the slope of $\ln(\cdot + m - 1)$ must then be strictly smaller than the slope of $g_m(\cdot)$. But we know from Theorem 11 that

$$\frac{\partial}{\partial \xi} \ln(\xi + m - 1) = \frac{1}{\xi + m - 1} \geq g_m'(\xi). \tag{75}$$

This is a contradiction which leads to the conclusion that there cannot be another crossing and $\ln(\cdot + m - 1)$ must be a strict lower bound to $g_m(\cdot)$.

Acknowledgment

This work was supported by the Industrial Technology Research Institute (ITRI), Zhudong, Taiwan, under JRC NCTU-ITRI and by the National Science Council under NSC 95-2221-E-009-046.

References

[1] A. Lapidoth and S. M. Moser, "The expected logarithm of a non-central chi-square random variable," website. [Online]. Available: http://moser.cm.nctu.edu.tw/explog.html

[2] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 5th ed., A. Jeffrey, Ed. Academic Press, San Diego, 1996.

[3] N. L. Johnson, S. Kotz, and N. Balakrishnan, *Continuous Univariate Distributions*, 2nd ed. John Wiley & Sons, 1995, vol. 2.

[4] A. Lapidoth and S. M. Moser, "Capacity bounds via duality with applications to multiple-antenna systems on flat fading channels," *IEEE Transactions on Information Theory*, vol. 49, no. 10, pp. 2426–2467, October 2003.

[5] S. M. Moser, "Duality-based bounds on channel capacity," Ph.D. dissertation, Swiss Federal Institute of Technology, Zurich, October 2004, Diss. ETH No. 15769. [Online]. Available: http://moser.cm.nctu.edu.tw/

[6] ——, "Fading number and degrees of freedom," March 2007, in preparation.

[7] ——, "The fading number of memoryless multiple-input multiple-output fading channels," July 2007, to appear in *IEEE Transactions on Information Theory*, vol. 53, no. 7. [Online]. Available: http://moser.cm.nctu.edu.tw/publications.html