The Fading Number of Multiple-Input Multiple-Output Fading Channels with Memory

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1 The Channel Model

We consider a multiple-input multiple-output (MIMO) fading channel:

$$\mathbf{Y}_k = \mathbb{H}_k \mathbf{x}_k + \mathbf{Z}_k$$

where

- $\mathbf{Y}_k \in \mathbb{C}^{n_\mathrm{R}}$ denotes the time-k channel output random vector;
- $\{\mathbf{Z}_k\}$ IID $\sim \mathcal{N}_\mathbb{C}\bigl(0,\sigma^2\bigr)$ denotes zero-mean, circularly symmetric, white, complex Gaussian noise of variance $\sigma^2 > 0$;
- $\mathbf{x}_k \in \mathbb{C}^{n_\mathrm{T}}$ denotes the time-k channel input vector satisfying either a peak-power or an average-power constraint $\mathcal{E}$ with signal-to-noise ratio $\textsc{snr} \triangleq \frac{\mathcal{E}}{\sigma^2}$;
- $\mathbb{H}_k$ denotes the time-k random $n_\mathrm{R} \times n_\mathrm{T}$ fading matrix of general (not necessarily Gaussian!) law including memory: we only assume that $\{\mathbb{H}_k\}$ is stationary, ergodic, of finite second moment $\mathsf{E}\bigl[\|\mathbb{H}_k\|_\mathrm{F}^2\bigr] < \infty$, and of finite differential entropy rate $h(\{\mathbb{H}_k\}) > -\infty$ (the *regularity* assumption). The realizations of $\{\mathbb{H}_k\}$ are *unknown* both at transmitter and receiver (*noncoherent situation*).

2 Channel Capacity

We know that for such regular, noncoherent fading channels the capacity grows double-logarithmically at very high power ($\textsc{snr} \to \infty$):

$$\mathsf{C}(\textsc{snr}) = \log(1+\log(1+\textsc{snr})) + \chi\bigl(\{\mathbb{H}_k\}\bigr) + o(1)$$

where $o(1)$ tends to zero as $\textsc{snr} \to \infty$ and where χ is a constant called **fading number** that is independent of the SNR.

*This work was supported by the Industrial Technology Research Institute (ITRI), Zhudong, Taiwan, under JRC NCTU-ITRI and by the National Science Council, Taiwan, under NSC 95-2221-E-009-046.

3 Main Results

Theorem 1. *Assume some general stationary channel model with input* $\mathbf{x}_k \in \mathbb{C}^{n_\mathrm{T}}$ *and output* $\mathbf{Y}_k \in \mathbb{C}^{n_\mathrm{R}}$. *Then under very mild conditions on the channel law and apart from edge effects the capacity-achieving input distribution can be assumed to be* stationary.

Lemma 2. *The capacity-achieving input process of the channel model in Section 1 can be assumed to be circularly symmetric,* i.e., *the input* $\{\mathbf{X}_k\}$ *can be replaced by* $\{\mathbf{X}_k e^{\mathrm{i}\Theta_k}\}$, *where* $\{\Theta_k\}$ *is IID* $\sim \mathcal{U}([0,2\pi])$ *and independent of every other random quantity.*

Theorem 3. *The MIMO fading number with memory* $\chi\bigl(\{\mathbb{H}_k\}\bigr)$ *is given by*

$$\begin{aligned}\chi\bigl(\{\mathbb{H}_k\}\bigr) &= \sup_{\substack{Q_{\{\hat{\mathbf{X}}_k\}}\\ \text{stationary}\\ \text{circ. sym.}}} \Biggl\{ h_\lambda\left(\frac{\mathbb{H}_0\hat{\mathbf{X}}_0}{\|\mathbb{H}_0\hat{\mathbf{X}}_0\|} \,\middle|\, \left\{\frac{\mathbb{H}_\ell\hat{\mathbf{X}}_\ell}{\|\mathbb{H}_\ell\hat{\mathbf{X}}_\ell\|}\right\}_{\ell=-\infty}^{-1} \right) \\ &\qquad + n_\mathrm{R}\mathsf{E}\Bigl[\log\|\mathbb{H}_0\hat{\mathbf{X}}_0\|^2\Bigr] - \log 2 \\ &\qquad - h\bigl(\mathbb{H}_0\hat{\mathbf{X}}_0 \bigm| \{\mathbb{H}_\ell\hat{\mathbf{X}}_\ell\}_{\ell=-\infty}^{-1}, \hat{\mathbf{X}}_{-\infty}^{0}\bigr) \Biggr\}\end{aligned}$$

where the maximization is over all stochastic unit-vector processes $\{\hat{\mathbf{X}}_k\}$ *that are stationary and circularly symmetric.*

Moreover, the fading number is achievable by a stationary input that can be expressed as a product of two independent *processes:* $\mathbf{X}_k = R_k \cdot \hat{\mathbf{X}}_k$. *Here* $\{\hat{\mathbf{X}}_k\} \in \mathbb{C}^{n_\mathrm{T}}$ *is a stationary and circularly symmetric unit-vector process with the probability distribution that maximizes the fading number, and* $\{R_k\} \in \mathbb{R}_0^+$ *is a scalar nonnegative IID random process such that*

$$\log R_k^2 \sim \mathcal{U}\left([\log\log\mathcal{E}, \log\mathcal{E}]\right).$$