The Fading Number of Memoryless Multiple-Input–Multiple-Output Fading Channels

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1 Channel Model

We consider a multiple-input–multiple-output fading channel:

$$\mathbf{Y} = \mathbb{H}\mathbf{x} + \mathbf{Z}$$

where

- $\mathbf{Y} \in \mathbb{C}^{n_\mathrm{R}}$ denotes the random channel output vector (n_R **receive antennas**);
- $\mathbf{Z} \sim \mathcal{N}_\mathbb{C}\left(\mathbf{0}, \sigma^2 \mathsf{I}_{n_\mathrm{R}}\right)$ denotes additive white (complex) **Gaussian noise**;
- $\mathbf{x} \in \mathbb{C}^{n_\mathrm{T}}$ denotes the channel input vector (n_T **transmit antennas**) satisfying either a peak-power or an average-power constraint $\mathcal{E}$ with signal-to-noise ratio $\textsc{snr} \triangleq \frac{\mathcal{E}}{\sigma^2}$;
- $\mathbb{H}$ denotes the random $n_\mathrm{R} \times n_\mathrm{T}$ fading matrix of **general** (not necessarily Gaussian!) **law**: we only assume that $\mathbb{H}$ has finite second moment $\mathsf{E}\left[\|\mathbb{H}\|_\mathrm{F}^2\right] < \infty$ and finite differential entropy $h(\mathbb{H}) > -\infty$ (the **regularity** assumption). Note that the components of $\mathbb{H}$ may be dependent (**spatial memory**). The realizations of $\mathbb{H}$ are **unknown** both at transmitter and receiver (**non-coherent decoding**).

2 Channel Capacity

We know that for such regular, non-coherent fading channels the capacity grows double-logarithmically at very high power ($\textsc{snr} \to \infty$):

$$\mathsf{C}(\textsc{snr}) = \log(1 + \log(1 + \textsc{snr})) + \chi(\mathbb{H}) + o(1)$$

where $o(1)$ tends to zero as $\textsc{snr} \to \infty$ and where $\chi(\mathbb{H})$ is a constant called **fading number** that is independent of $\textsc{snr}$, but depends on the distribution of $\mathbb{H}$.

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3 Main Results

Lemma 1. *Any capacity-achieving input distribution must* ***escape to infinity****. This means that for $\textsc{snr} \to \infty$ the capacity-achieving input distributions will not use any finite-cost input symbols.*

Lemma 2. *The capacity-achieving input can be assumed to be* ***circularly symmetric****,* i.e., *the input $\mathbf{X}$ can be replaced by $\mathbf{X}e^{i\Theta}$, where $\Theta \sim \mathcal{U}([0, 2\pi])$ and is independent of every other random quantity.*

Theorem 3. *The memoryless mulitple-input multiple-output (MIMO) fading number $\chi(\mathbb{H})$ is given by*

$$\chi(\mathbb{H}) = \sup_{Q_{\hat{\mathbf{X}}}} \left\{ h_\lambda\left(\frac{\mathbb{H}\hat{\mathbf{X}}}{\|\mathbb{H}\hat{\mathbf{X}}\|}\right) + n_\mathrm{R}\mathsf{E}\left[\log \|\mathbb{H}\hat{\mathbf{X}}\|^2\right] - \log 2 - h\left(\mathbb{H}\hat{\mathbf{X}} \,\middle|\, \hat{\mathbf{X}}\right) \right\}$$

where the supremum is taken over all distributions of the ***random unit vector*** $\hat{\mathbf{X}}$*. Note that $h_\lambda(\cdot)$ denotes a differential entropy for random vectors that take value on the unit sphere in $\mathbb{C}^{n_\mathrm{R}}$.*

Moreover, this fading number is ***achievable by*** *a random vector $\mathbf{X} = \hat{\mathbf{X}} \cdot R$ where $\hat{\mathbf{X}}$ is distributed according to the distribution that achieves the fading number above and where R is a non-negative random variable independent of $\hat{\mathbf{X}}$ such that*

$$\log R^2 \sim \mathcal{U}([\log\log\mathcal{E}, \log\mathcal{E}]) \,.$$

References

[1] S. M. Moser, "The fading number of memoryless multiple-input multiple-output fading channels," *IEEE Transactions on Information Theory*, vol. 53, no. 7, pp. 2652–2666, July 2007.