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Expectations of a Noncentral Chi-Square Distribution With Application to IID MIMO Gaussian Fading

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Abstract

In this paper closed-form expressions are derived for the expectation of the logarithm and for the expectation of the n-th power of the reciprocal value (inverse moments) of a noncentral chi-square random variable of even degree of freedom. It is shown that these expectations can be expressed by a family of continuous functions $g_m(\cdot)$ and that these families have nice properties (monotonicity, convexity, *etc.*). Moreover, some tight upper and lower bounds are derived that are helpful in situations where the closed-form expression of $g_m(\cdot)$ is too complex for further analysis.

As an example of the applicability of these results, in the second part of this paper an independent and identically distributed (IID) Gaussian multiple-input–multiple-output (MIMO) fading channel with a scalar line-of-sight component is analyzed. Some new expressions are derived for the fading number that describes the asymptotic channel capacity at high signal-to-noise ratios (SNR).

1. INTRODUCTION

It is well known that adding several independent squared Gaussian random variables of identical variance, but possibly different mean yields a random variable that is noncentral chi-square distributed. Since Gaussian random variables are fundamentally important and are used in a wide variety of fields (like, *e.g.*, statistical theory, information theory, or communications), it is not surprising that the noncentral chi-square distribution often shows up in science and engineering [1]. It not only includes the important cases of a squared Rayleigh distribution and a squared Rice distribution, but also the generalizations to a sum of independent squared Gaussian random variables of identical variance with or without mean, *i.e.*, a "squared multiple Rayleigh" and "squared multiple Rice" distribution.

In this paper we will present some new closed-form expressions for different expectations of a noncentral chi-squared distributed random variable V of even degree of freedom. (Note that in practice we often have an even number of degrees of freedom because usually *complex* Gaussian random variables consisting of *two real* Gaussian components are considered.) In particular we will state new expressions for the expected value of the logarithm of V, $\mathsf{E}[\ln V]$, and of the inverse moments of V, $\mathsf{E}[V^{-n}]$. While for the inverse moments exact expressions as a finite sum are already known [2] [1], the new expressions that are presented here show the connections between these expectations and are presented based on a single function $g_m(\cdot)$ (and its derivatives). Hence, our expressions turn out to be pleasingly simple. The expression for the expected value of the logarithm of V and its connection to the inverse moments are—to our knowledge—new.

Moreover we present some fundamental properties of the function $g_m(\cdot)$ and derive very tight upper and lower bounds. This will simplify working with these results in situations where the exact expression is too complicated to deal with. Some other known results as, *e.g.*, the asymptotics of the first inverse moment [3], follow directly and easily from our results.

In the second part of this paper we will then present a simple example where the derived results about the noncentral chi-square distribution are applied. We will consider the situation of a noncoherent[1] multiple-input multiple-output (MIMO) Gaussian fading channel and derive some new results concerning its *channel capacity.* The channel capacity is defined as the ultimate communication rate for which *reliable* communication is possible, *i.e.*, for which at least theoretically it is

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[1] *I.e.*, the realization of the multiplicative fading noise is not *a priori* known at the receiver or transmitter.

possible to communicate with an arbitrary small error probability.

We will derive new results concerning the *fading number* of such Gaussian fading channels. The fading number is the second term in the high-SNR expansion of capacity and has been introduced in [4]. For a discussion of its practical meaning we refer to [5].

The outline of this paper is as follows. In the subsequent section we will define the noncentral chi-square distribution and state our main results about the expected logarithm and the inverse moments. In Section 3 some properties and bounds on the family of functions $g_m(\cdot)$ are presented. And Section 4 then applies these results to a Gaussian fading channel.

2. EXPECTATIONS OF A NONCENTRAL CHI-SQUARE DISTRIBUTION

For brevity we will omit all proofs. They can be found in [6].

A nonnegative real random variable $W_{(r),\xi}$ is said to have a *noncentral chi-square* distribution with r *degrees of freedom* and *noncentrality parameter* $\xi \geq 0$ if it is distributed like

$$W_{(r),\xi} \triangleq \sum_{j=1}^{r} (X_j + \mu_j)^2, \tag{1}$$

where $\{X_j\}_{j=1}^{r}$ are independent and identically distributed (IID) $\sim \mathcal{N}_{\mathbb{R}}(0,1)$ and the real constants $\{\mu_j\}_{j=1}^{r}$ satisfy $\xi = \sum_{j=1}^{r} \mu_j^2$. (The distribution of (1) depends on the constants $\{\mu_j\}$ only via the sum of their squares.) The probability density function of such a distribution is given by [1, Chapter 29]

$$f_{W_{(r),\xi}}(x) = \frac{1}{2}\left(\frac{x}{\xi}\right)^{\frac{r-2}{4}} e^{-\frac{\xi+x}{2}} I_{r/2-1}\left(\sqrt{\xi x}\right), \quad x \geq 0. \tag{2}$$

Here $I_\nu(\cdot)$ denotes the modified Bessel function of the first kind of order $\nu \in \mathbb{R}$ [7, Eq. 8.445].

If the number of degrees of freedom r is even, *i.e.*, if $r = 2m$ for some positive integer m, then the noncentral chi-square distribution can also be expressed as a sum of the squared norms of *complex* Gaussian random variables. To see this recall that a complex Gaussian random variable consists of two independent real Gaussian random variable as real and imaginary part, respectively. However, note that the variance of these real Gaussian random variables is only half the variance of the corresponding complex Gaussian random variable, *i.e.*, for $U \sim \mathcal{N}_{\mathbb{C}}(0,1)$ with

$$U = X_{\text{re}} + \mathrm{i} X_{\text{im}} \tag{3}$$

we have $X_{\text{re}}, X_{\text{im}} \sim \mathcal{N}_{\mathbb{R}}\big(0, \frac{1}{2}\big)$. Hence, our engineering type definition of a noncentral chi-square distribution will differ by some factors of 2 from the common definition in the mathematical literature [1] (see also (6)).

Definition 1. *We say that the random variable $V_{(m),\xi}$ has a noncentral chi-square distribution with an even number $2m$ of degrees of freedom if*

$$V_{(m),\xi} \triangleq \sum_{j=1}^{m} \left|U_j + \mu_j\right|^2 \tag{4}$$

where $\{U_j\}_{j=1}^{m}$ are IID $\sim \mathcal{N}_{\mathbb{C}}(0,1)$, $\{\mu_j\}_{j=1}^{m}$ are complex constants, and where the noncentrality parameter ξ is defined as

$$\xi \triangleq \sum_{j=1}^{m} |\mu_j|^2. \tag{5}$$

Note the following relation to the other definition of a noncentral chi-square distribution (1):

$$V_{(m),\xi} = \frac{1}{2} W_{(2m),2\xi}. \tag{6}$$

Next we define the following family of continuous functions.

Definition 2. *The functions $g_m(\cdot)$ are defined as follows:*

$$g_m(\xi) \triangleq \begin{cases} \ln(\xi) - \mathrm{Ei}(-\xi) + \displaystyle\sum_{j=1}^{m-1} (-1)^j \Bigg[e^{-\xi}(j-1)! \\ \qquad - \dfrac{(m-1)!}{j(m-1-j)!} \Bigg] \left(\dfrac{1}{\xi}\right)^j, & \xi > 0 \\ \psi(m), & \xi = 0 \end{cases} \tag{7}$$

for $m \in \mathbb{N}$, where $\mathrm{Ei}(\cdot)$ denotes the exponential integral function defined as

$$\mathrm{Ei}(-\xi) \triangleq -\int_{\xi}^{\infty} \frac{e^{-t}}{t}\, \mathrm{d}t, \qquad \xi > 0 \tag{8}$$

and $\psi(\cdot)$ is Euler's psi function given by

$$\psi(m) \triangleq -\gamma + \sum_{j=1}^{m-1} \frac{1}{j}, \qquad m \in \mathbb{N} \tag{9}$$

with $\gamma \approx 0.577$ denoting Euler's constant.

Note that $g_m(\cdot)$ is continuous over $[0,\infty)$ for all $m \in \mathbb{N}$, *i.e.*, in particular

$$\lim_{\xi \downarrow 0} \left\{ \ln(\xi) - \mathrm{Ei}(-\xi) + \sum_{j=1}^{m-1} (-1)^j \Bigg[e^{-\xi}(j-1)! \right.$$

$$- \frac{(m-1)!}{j(m-1-j)!}\Bigg] \left(\frac{1}{\xi}\right)^j \Bigg\} = \psi(m) \quad (10)$$

for all $m \in \mathbb{N}$. Therefore its first derivative is defined[2] for all $\xi \geq 0$ and can be evaluated to

$$g'_m(\xi) \triangleq \frac{\partial g_m(\xi)}{\partial \xi} = \frac{(-1)^m \Gamma(m)}{\xi^m} \left(e^{-\xi} - \sum_{j=0}^{m-1} \frac{(-1)^j}{j!} \xi^j \right) \quad (11)$$

(see [4, Eq. (417)], [8, Eq. (A.39)]). Note that $g'_m(\cdot)$ is also continuous, *i.e.*, in particular

$$\lim_{\xi \downarrow 0} \left\{ \frac{(-1)^m \Gamma(m)}{\xi^m} \left(e^{-\xi} - \sum_{j=0}^{m-1} \frac{(-1)^j}{j!} \xi^j \right) \right\} = \frac{1}{m} = g'_m(0). \quad (12)$$

Now we will give closed-form expressions for some expectations of a noncentral chi-square random variable. We start with the logarithm.

Theorem 3. *The expected value of the logarithm of a noncentral chi-square random variable with an even number $2m$ of degrees of freedom is given as*

$$\boxed{\mathsf{E}\left[\ln V_{(m),\xi}\right] = g_m(\xi)} \quad (13)$$

where $V_{(m),\xi}$ and ξ are defined in (4) *and* (5).

Next we look at the inverse moments.

Theorem 4. *Let $n \in \mathbb{N}$ with $n < m$. The expected value of the n-th power reciprocal value of a noncentral chi-square random variable with an even number $2m$ of degrees of freedom is given as*

$$\boxed{\mathsf{E}\left[\left(\frac{1}{V_{(m),\xi}}\right)^n\right] = \frac{(-1)^{n-1}}{(n-1)!} \cdot g_{m-n}^{(n)}(\xi), \quad m > n} \quad (14)$$

where $g_m^{(n)}(\xi) \triangleq \frac{\partial^n g_m(\xi)}{\partial \xi^n}$ denotes the n-th derivative of $g_m(\cdot)$ and where $V_{(m),\xi}$ and ξ are defined in (4) *and* (5). *In particular, for $m > 1$*

$$\boxed{\mathsf{E}\left[\frac{1}{V_{(m),\xi}}\right] = g'_{m-1}(\xi).} \quad (15)$$

Note that in the cases where $m \leq n$ the expectation is unbounded.

[2] As a matter of fact all derivative of $g_m(\cdot)$ are defined and continuous.

3. PROPERTIES AND BOUNDS OF $g_m(\cdot)$ AND $g'_m(\cdot)$

Next we will present some properties and bounds of $g_m(\cdot)$ and $g'_m(\cdot)$.

Theorem 5. *The functions $g_m(\cdot)$ are monotonically strictly increasing and strictly concave in the interval $[0, \infty)$ for all $m \in \mathbb{N}$. The functions $g_m(\xi)$ are monotonically strictly increasing in m for all $\xi \geq 0$.*

The functions $g_m\left(\frac{1}{\xi}\right)$ are monotonically strictly decreasing and convex in ξ for all $m \in \mathbb{N}$.

The functions $g'_m(\cdot)$ are positive, monotonically strictly decreasing, and strictly convex functions for all $m \in \mathbb{N}$. The functions $g'_m(\xi)$ are monotonically strictly decreasing in m for all $\xi \geq 0$.

We have the following relations:

$$g_{m+1}(\xi) = g_m(\xi) + g'_m(\xi) \quad (16)$$

for all $m \in \mathbb{N}$ and all $\xi \geq 0$, and

$$g'_{m+1}(\xi) = \frac{1}{\xi} - \frac{m}{\xi} g'_m(\xi), \qquad \xi > 0, \quad (17)$$

$$\text{and} \qquad g'_m(\xi) = \frac{1}{m} - \frac{\xi}{m} g'_{m+1}(\xi), \quad \xi \geq 0, \quad (18)$$

for all $m \in \mathbb{N}$.

Note that relation (16) nicely reflects a property discovered by Cohen [9] [1, Eq. (29.23a)].

Next we state some bounds.

Theorem 6. *The function $g'_m(\cdot)$ can be bounded as follows:*

$$\frac{1}{\xi + m} \leq g'_m(\xi) \leq \min\left\{ \frac{m+1}{m(\xi+m+1)}, \frac{1}{\xi+m-1} \right\}. \quad (19)$$

Note that for $\xi < m+1$ the first of the two upper bounds is tighter than second, while for $\xi > m+1$ the second is tighter. Moreover, the first upper bound coincides with $g_m(\xi)$ for $\xi = 0$, and the second upper bound is asymptotically tight when ξ tends to infinity.

For the functions $g_m(\cdot)$ we state two sets of bounds. The first set is tighter:

$$g_m(\xi) \geq \ln \xi - \mathrm{Ei}\left(-\xi\right) + \sum_{j=1}^{m-1} \frac{1}{\xi + j}, \quad (20)$$

$$g_m(\xi) \leq \ln \xi - \mathrm{Ei}\left(-\xi\right) + \sum_{j=1}^{m-1} \min\left\{ \frac{j+1}{j(\xi+j+1)}, \frac{1}{\xi+j-1} \right\}. \quad (21)$$

Secondly, we give a set of bounds that is slightly less tight, but that is very appealing because the expressions are simple and easy to use for further analysis:

$$\ln(\xi + m - 1) \leq g_m(\xi) \leq \ln(\xi + m). \quad (22)$$

The bounds (19) are depicted in Figure 1, the bounds (20) and (21) in Figure 2 and the bounds (22) are depicted in Figure 3. In all three cases the bounds are shown for the values $m = 1$, $m = 3$, and $m = 8$.

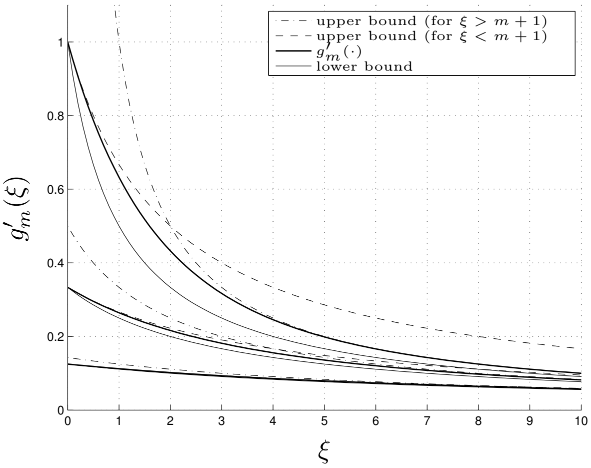

Figure 1: Upper and lower bounds on $g'_m(\cdot)$ according to (19) in Theorem 6. The top four curves correspond to $m = 1$, the middle four to $m = 3$, and the lowest group of four curves to $m = 8$.

Note that some of the asymptotic results given in [3] can be directly derived from (15) and (19).

We also remark that the known expression [2] [1, Eq. (29.32a)]

$$\mathsf{E}\left[\left(\frac{1}{V_{(m),\xi}}\right)^n\right] = \frac{(-1)^{n+m}}{(n-1)!}\sum_{\ell=0}^{n-1}\binom{n-1}{\ell}\xi^{\ell+1-m} \cdot \Gamma(m-\ell-1)\left(e^{-\xi} - \sum_{j=0}^{m-\ell-2}\frac{(-1)^j}{j!}\xi^j\right) \quad (23)$$

can be used as an alternative way of finding the n-th derivative of $g_m(\cdot)$:

$$g_m^{(n)}(\xi) = (-1)^{m+n-1}\sum_{\ell=0}^{n-1}\binom{n-1}{\ell}\xi^{-(m-\ell)-(n-1)} \cdot \Gamma\big(m-\ell-(n-1)\big)\left(e^{-\xi} - \sum_{j=0}^{m-\ell+n-2}\frac{(-1)^j}{j!}\xi^j\right). \quad (24)$$

Finally, we also state another expression for the derivatives of $g_m(\cdot)$ that uses an infinite sum [4] [1,

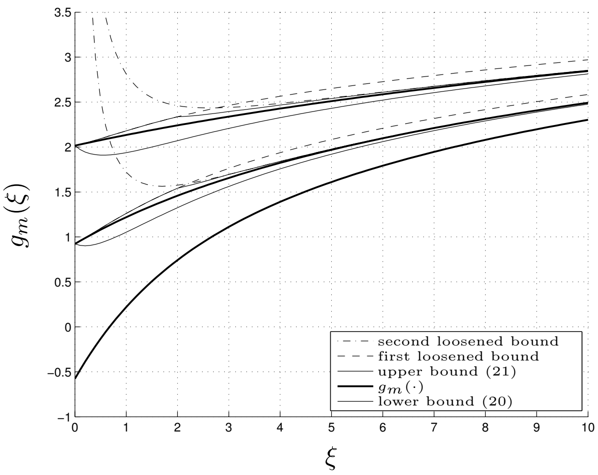

Figure 2: Upper and lower bounds on $g_m(\cdot)$ according to (20) and (21) in Theorem 6. We also depict the two additional upper bounds that are gained by loosening the minimum in (21) to always take on one of the two values. The lowest curve corresponds to $m = 1$ (in this case all bounds coincides with $g_1(\cdot)$), the next five curves correspond to $m = 3$, and the top five to $m = 8$.

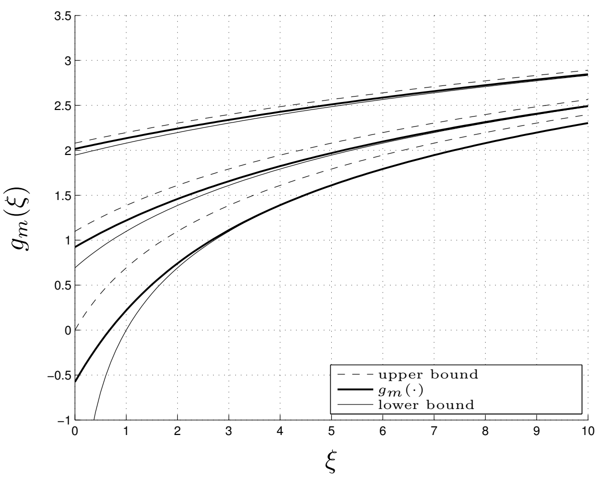

Figure 3: Upper and lower bounds on $g_m(\cdot)$ according to (22) in Theorem 6. The lowest three curves correspond to $m = 1$, the next three to $m = 3$, and the top three to $m = 8$.

Eq. (29.32c)]:

$$g_m^{(n)}(\xi) = e^{-\xi} \sum_{k=0}^{\infty} \frac{(-1)^{n-1}(n-1)!}{(k+m)\cdots(k+m+n-1)} \cdot \frac{\xi^k}{k!}. \tag{25}$$

4. FADING NUMBER OF AN IID GAUSSIAN FADING CHANNEL WITH A SCALAR LINE-OF-SIGHT COMPONENT

In this section a small example is shown where some of the results from Sections 2 and 3 can be applied. To that goal we consider the following Gaussian MIMO fading channel:

$$\mathbf{Y} = \mathbb{H}\mathbf{x} + \mathbf{Z}. \tag{26}$$

Here the n_R-vector $\mathbf{Y}$ denotes the signal at the receiver, the n_T-vector $\mathbf{x}$ denotes the transmitted signal, $\mathbf{Z}$ is additive white Gaussian noise, and the $n_\mathrm{R} \times n_\mathrm{T}$ fading matrix $\mathbb{H}$ corresponds to multiplicative noise. It is assumed that $\mathbb{H}$ can be written as

$$\mathbb{H} = \mathsf{D} + \tilde{\mathbb{H}} \tag{27}$$

where all components of the $n_\mathrm{R} \times n_\mathrm{T}$ random matrix $\tilde{\mathbb{H}}$ are IID $\sim \mathcal{N}_\mathbb{C}(0,1)$ and where the constant $n_\mathrm{R} \times n_\mathrm{T}$ line-of-sight matrix D is scalar in the sense that for some constant $d \in \mathbb{C}$ we have in the case $n_\mathrm{R} \le n_\mathrm{T}$:

$$\mathsf{D} = d \left(\mathsf{I}_{n_\mathrm{R}} \quad \mathsf{0}_{n_\mathrm{R}\times(n_\mathrm{T}-n_\mathrm{R})} \right) \tag{28}$$

and in the case $n_\mathrm{R} > n_\mathrm{T}$:

$$\mathsf{D} = d \begin{pmatrix} \mathsf{I}_{n_\mathrm{T}} \\ \mathsf{0}_{(n_\mathrm{R}-n_\mathrm{T})\times n_\mathrm{T}} \end{pmatrix}, \tag{29}$$

where I_n denotes the $n \times n$ identity matrix and $\mathsf{0}_{n\times m}$ is the $n \times m$ zero-matrix.

Note that conditional on the input $\mathbf{x}$ the squared magnitude of the output $\|\mathbf{Y}\|^2$ is noncentral chi-square distributed.

For this fading channel we would like to derive the asymptotic capacity in the case when the available SNR tends to infinity. In particular we are interested in the *fading number* which is the second term in the high-SNR expansion of capacity and has been introduced in [4]:

$$\chi(\mathbb{H}) \triangleq \varlimsup_{\mathrm{SNR}\uparrow\infty} \left\{ \mathsf{C}(\mathrm{SNR}) - \log(1+\log(1+\mathrm{SNR})) \right\}. \tag{30}$$

An expression for the general memoryless MIMO fading number has already been derived in [10]:

$$\chi(\mathbb{H}) = \sup_{Q_{\hat{\mathbf{X}}}} \left\{ h_\lambda\left(\frac{\mathbb{H}\hat{\mathbf{X}}}{\|\mathbb{H}\hat{\mathbf{X}}\|} \right) + n_\mathrm{R} \mathsf{E}\left[\log \|\mathbb{H}\hat{\mathbf{X}}\|^2 \right] - \log 2 - h\left(\mathbb{H}\hat{\mathbf{X}} \,\middle|\, \hat{\mathbf{X}}\right) \right\}, \tag{31}$$

however, this expression still contains an optimization. Hence, we would like to evaluate (31) in the situation of the IID MIMO Gaussian fading channel with a scalar line-of-sight component (27)–(29).

Again we omit all proofs.

Theorem 7. *Assume $n_\mathrm{R} \le n_\mathrm{T}$ and a Gaussian fading matrix as given in* (27) *and* (28)*. Then*

$$\chi(\mathbb{H}) = n_\mathrm{R} g_{n_\mathrm{R}}\left(|d|^2\right) - n_\mathrm{R} - \log\Gamma(n_\mathrm{R}) \tag{32}$$

where $g_m(\cdot)$ is defined in (7)*.*

For the case $n_\mathrm{R} > n_\mathrm{T}$ we firstly need to introduce some notation. We note that

$$\mathbb{H}\hat{\mathbf{X}} = \mathsf{D}\hat{\mathbf{X}} + \tilde{\mathbb{H}}\hat{\mathbf{X}} \overset{\mathscr{L}}{=} \begin{pmatrix} d\hat{\mathbf{X}} \\ \mathbf{0} \end{pmatrix} + \tilde{\mathbf{H}}, \tag{33}$$

where $\tilde{\mathbf{H}} \sim \mathcal{N}_\mathbb{C}(\mathbf{0}, \mathsf{I}_{n_\mathrm{R}})$, and we split the vector $\tilde{\mathbf{H}}$ into two parts:

$$\tilde{\mathbf{H}} = \begin{pmatrix} \tilde{\mathbf{H}}_1 \\ \tilde{\mathbf{H}}_2 \end{pmatrix} \tag{34}$$

where $\tilde{\mathbf{H}}_1 \sim \mathcal{N}_\mathbb{C}\left(\mathbf{0}, \mathsf{I}_{n_\mathrm{T}}\right)$ and $\tilde{\mathbf{H}}_2 \sim \mathcal{N}_\mathbb{C}\left(\mathbf{0}, \mathsf{I}_{n_\mathrm{R}-n_\mathrm{T}}\right)$ are two independent white Gaussian random vectors in $\mathbb{C}^{n_\mathrm{T}}$ and $\mathbb{C}^{n_\mathrm{R}-n_\mathrm{T}}$, respectively. Then we write

$$\mathbb{H}\hat{\mathbf{X}} \overset{\mathscr{L}}{=} \begin{pmatrix} d\hat{\mathbf{X}} + \tilde{\mathbf{H}}_1 \\ \tilde{\mathbf{H}}_2 \end{pmatrix} \tag{35}$$

and define

$$S_1 \triangleq \left\| d\hat{\mathbf{X}} + \tilde{\mathbf{H}}_1 \right\|^2, \tag{36}$$

$$S_2 \triangleq \left\| \tilde{\mathbf{H}}_2 \right\|^2. \tag{37}$$

Note that S_1 is noncentral chi-square distributed with $2n_\mathrm{T}$ degrees of freedom and noncentrality parameter $\|d\hat{\mathbf{X}}\|^2 = |d|^2$ independently of the distribution of $\hat{\mathbf{X}}$, and that S_2 is central chi-square distributed with $2(n_\mathrm{R} - n_\mathrm{T})$ degrees of freedom. Moreover, S_1 and S_2 are independent of each other.

Theorem 8. *Assume $n_\mathrm{R} > n_\mathrm{T}$ and a Gaussian fading matrix as given in* (27) *and* (29)*. Then*

$$\chi(\mathbb{H}) = n_\mathrm{T} g_{n_\mathrm{T}}\left(|d|^2\right) - n_\mathrm{T} - \log\Gamma(n_\mathrm{T}) + I\left(S_1; \frac{S_2}{S_1} \right). \tag{38}$$

Unfortunately, we have not succeeded in deriving the term $I(S_1; S_2/S_1)$ precisely. However, we can state the asymptotic behavior when d gets large.

Corollary 9. *The fading number of the IID MIMO Gaussian fading channel as defined in* (27)–(29) *is given by*

$$\chi(\mathbb{H}) = n_{\mathrm{m}} g_{n_{\mathrm{m}}}\bigl(|d|^2\bigr) - n_{\mathrm{m}} - \log\Gamma\bigl(n_{\mathrm{m}}\bigr) + f(n_{\mathrm{R}}, n_{\mathrm{T}}; d) \tag{39}$$

where

$$n_{\mathrm{m}} \triangleq \min\{n_{\mathrm{R}}, n_{\mathrm{T}}\} \tag{40}$$

is called degree of freedom *of a MIMO fading channel and where* $f(n_{\mathrm{R}}, n_{\mathrm{T}}; d)$ *depends primarily on* n_{T} *and* n_{R} *and is bounded in* d*:*

$$\begin{aligned} 0 \le f(n_{\mathrm{R}}, n_{\mathrm{T}}; d) &\le (n_{\mathrm{R}} - n_{\mathrm{T}} - 1)\psi(n_{\mathrm{R}} - n_{\mathrm{T}}) \\ &\quad - (n_{\mathrm{R}} - n_{\mathrm{T}} - 1) - \log\frac{\Gamma(n_{\mathrm{R}} - n_{\mathrm{T}})}{n_{\mathrm{R}} - n_{\mathrm{T}}} \\ &\quad + \log\left(\min\left\{\frac{n_{\mathrm{T}}}{n_{\mathrm{T}} - 1}, \frac{\pi e}{2}\right\}\right). \end{aligned} \tag{41}$$

Using (22) *we therefore have for* $|d| \gg 1$

$$\chi \sim n_{\mathrm{m}} \log\bigl(|d|^2 + n_{\mathrm{m}}\bigr). \tag{42}$$

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